AD-A115 754 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 12/1
EXTENSION OF SOME MODELS FOR POSITIVE-VALUED TIME SERIES.(U)
MAR 82 D K HUGUS

UNCLASSIFIED NL

1 OF 4
AD A
115754

(2)

NAVAL POSTGRADUATE SCHOOL
Monterey, California

AD A115754

THESIS

EXTENSION OF SOME MODELS FOR
POSITIVE-VALUED TIME SERIES

BY

David Kennedy Hugus

March 1982

Thesis Advisor: P. A. W. Lewis

Approved for public release; distribution unlimited.

036

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO. AD-A115 764

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
Extensions of Some Models for Positive-Valued Time Series

5. TYPE OF REPORT & PERIOD COVERED
Ph.D. Thesis; March 1982

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
David Kennedy Hugus

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Naval Postgraduate School
Monterey, California 93940

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Naval Postgraduate School
Monterey, California 93940

12. REPORT DATE
March 1982

13. NUMBER OF PAGES
356

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Autoregressive; Moving Average; Mixed Autoregressive-Moving Average; Gamma Marginals; Exponential Marginals; Negative Correlation; Antithetic Variables; Maximum Likelihood Estimation; GLARMA(1,1); GLAR(2); NEMA(1); GLAR(1); EMA(1); Wind Speed Data Analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Time series models with autoregressive, moving average and mixed autoregressive-moving average correlation structure and with positive-valued non-normal marginal distributions are considered. First a flexible mixed model GLARMA(p,q) with Gamma marginals is investigated. The correlation structure for several special cases is derived. For the first-order autoregressive case, GLAR(1), the conditional density of X_n given X_{n-1}

(20. ABSTRACT Continued)

is derived. This leads to the formation of a likelihood function and a numerical approximation to and a simulation study of the maximum likelihood method of parameter estimation. Multivariate extensions of the model are considered briefly.

Second, three methods for generating first-order moving average sequences with Exponential marginals are examined. These generalize the EMA(1) Exponential model. Negative correlation using antithetic variables is investigated in the moving average models.

A preliminary analysis of wind speed data obtained over a 15 year period in the Gulf of Alaska is presented. A model with four harmonic deterministic mean multiplying random innovative factors modeled by a GLAR(1) process is developed. Correlograms and periodograms are used to determine the model for the mean and the structure of the innovation process.

Approved for public release; distribution unlimited.

EXTENSION OF SOME MODELS FOR
POSITIVE-VALUED TIME SERIES

by

David Kennedy Hugus
Lieutenant Colonel, United States Army
B.S., Purdue University, 1966
M.S., Naval Postgraduate School, 1973

Submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

from the

NAVAL POSTGRADUATE SCHOOL
March 1982

Author: David K Hugus

Approved by:

Peter A. W. Lewis
P. A. W. LEWIS, Professor of
Operations Research and Statistics
DISSERTATION SUPERVISOR

James K Hartman
J. K. HARTMAN
Associate Professor of
Operations Research

F. R. Richards
F. R. RICHARDS
Associate Professor of
Operations Research

G. H. BRADLEY, Professor and
Chairman of Computer Science

R. J. RENARD, Professor and
Chairman of Meteorology

R R Read
R. R. READ, Professor of
Operations Research

Approved by: ______________________
Chairman, Department of Operations Research

Approved by: ______________________
Academic Dean

ABSTRACT

Time series models with autoregressive, moving average and mixed autoregressive-moving average correlation structure and with positive-valued non-normal marginal distributions are considered. First, a flexible mixed model GLARMA(p,q) with Gamma marginals is investigated. The correlation structure for several special cases is derived. For the first-order autoregressive case, GLAR(1), the conditional density of X_n given X_{n-1} is derived. This leads to the formation of a likelihood function and a numerical approximation to and a simulation study of the maximum likelihood method of parameter estimation. Multivariate extensions of the model are considered briefly.

Second, three methods for generating first-order moving average sequences with Exponential marginals are examined. These generalize the EMA(1) Exponential model. Negative correlation using antithetic variables is investigated in the moving average models.

A preliminary analysis of wind speed data obtained over a 15 year period in the Gulf of Alaska is presented. A model with four harmonic deterministic mean multiplying random innovative factors modeled by a GLAR(1) process is developed. Correlograms and periodograms are used to determine the model for the mean and the structure of the innovation process.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENT

I want to take this opportunity to express my gratitude to the following individuals:

Professor P. A. W. Lewis whose efficiency in getting my research started and whose unflagging enthusiasm and support are largely responsible for getting this project completed.

Professor R. R. Read whose counsel and personal support were as valuable as his technical knowledge when I was preparing for my qualifying exams.

Associate Professor J. K. Hartman who helped even when he couldn't afford the time and whose suggestion concerning the search technique for the maximum likelihood program saved many hours of CPU time.

Associate Professor F. R. Richards who filled the gap with virtually no notice.

Professors G. H. Bradley and R. J. Renard who took time from busy schedules to help an outsider.

Professor D. R. Barr whose wise advice may have made the difference between success and failure.

Distinguished Professor Emiritus F. D. Faulkner and Associate Professor R. H. Franke who showed me how to eliminate the singularities in a conditional density.

Assistant Prcfessor R. W. Garwood who provided the data for analysis and lent his oceanographical expertise to the evaluation of the analysis.

Ladies and Gentlemen of W. R. Church Computer Center, especially those on the 1600-2400 shift, whose support was invaluable during periods when plotter and CPU use were intense.

Nancy Ritz Hugus whose support and understanding make all good things possible.

I. INTRODUCTION

The classical approach to time series analysis based on linear, additive models with normally distributed, constant variance residuals is probably best presented by the work of Box and Jenkins [Ref. 1]. Although their work is widely accepted and used, it is not applicable to some important time series. This is mainly because the Box-Jenkins approach is based on an assumed normal distribution for the series in question. However, the assumption of normality is not appropriate when the series is known to be non-negative. Such series typically involve times between successive events in event processes. Examples are easy to construct. Times between arrivals at a hospital emergency room, times between breakdowns in a tank main drive assembly, and times between detections of enemy armor vehicles are a sample of series of this type. Because of the non-negative nature of the series, the Box-Jenkins distributional assumptions and, hence, the analysis techniques are inappropriate. There is, of course, the possibility of data transformations but this is not appropriate with very skewed marginal distributions and it is, in most cases, difficult to ascertain what the transformation does to the correlation structure of the series.

Gaver and Lewis [Ref. 2] wrote the pioneering paper on the subject of autoregressive processes with non-normal marginal distributions. They presented the method for determining the

distribution of the innovative terms in the basic, linear, additive, autoregressive equations (first-order stochastic different equation)

$$X_n = \rho X_{n-1} + \varepsilon_n \qquad \text{(I.1)}$$

that was required to produce a given marginal distribution for the $\{X_n\}$ sequence. They presented results for $\{X_n\}$ sequences with Exponential, Gamma, and mixed Exponential marginals. They also showed that this problem was the same as that of determining the class of self-decomposable (Type L) random variables (Feller, [Ref. 3], Loeve [Ref. 4]) although the connection between the solution to the self-decomposable problem and equation (I.1) was not explicit in the literature.

The Gaver and Lewis paper was followed by other papers which extended these results. Lawrance and Lewis [Ref. 5] presented a first-order moving average process with Exponential marginals. Jacobs and Lewis [Ref. 6] propounded a mixed autoregressive-moving average of order one, EARMA(1,1), and Exponential marginals. This was extended to an arbitrary order EARMA(p,q) process by Lawrance and Lewis [Ref. 7]. A further refinement of the first-order, Exponential, autoregressive process (NEAR(1)) was presented by Lawrance and Lewis [Ref. 8]. While this contained the previous EAR(1) model, it did not suffer from the degeneracy inherent in (I.1). Jacbos applied these models to closed cyclic queueing networks

[Ref. 9] and Lewis and Shedler applied them to models of computer processes [Ref. 10].

This paper extends the results of these researchers and others in three areas. In Chapter II a mixed ARMA(p,q) model with Gamma marginals proposed by Lewis [Ref. 11], the GLARMA(p,q) model, is examined. The correlation structure is derived for several values of p and q. Of particular note is the AR(1) case (p = 1, q = 0), called GLAR(1), where the conditional density of X_n given X_{n-1} is derived. This leads to the derivation of a likelihood function and a numerical technique to evaluate and maximize the likelihood function with respect to the model parameters. This provides a useful technique for estimating model parameters. Using this numerical technique a simulation study of the properties of maximum likelihood estimators for the parameters of the model is given.

The correlation structure is derived for other models in the GLARMA(p,q) family: the first-order moving average, the second-order autoregressive, the first-order mixed autoregressive-moving average and a bivariate first-order autoregressive process. These different models, particularly the bivariate extension, demonstrate the flexibility of the GLARMA(p,q) model.

In Chapter III the first-order moving average process with Exponential marginals of Lawrance and Lewis [Ref. 5] is extended to a two parameter model. This is done by utilizing the NEAR(1) structure which combines two independent Exponential random variables into a random variable with Exponential

distribution. A fairly complete set of characteristics of this model are derived. In particular the correlation structure, the quantity $P(X_{n+1} > X_n)$, the Laplace transform of sums of X_n's, the Laplace transforms of the distribution of counts, the (Bartlett) spectrum of counts, and the joint Laplace-Stieltjes transforms of X_n and X_{n+1} are addressed. These characteristics are compared to those of other processes which produce marginally Exponential random variables.

In Chapter IV the models of Chapter II are used in a preliminary data analysis of wind speed data. This represents the first effort to apply these models to a large, real world data base. A model for simulating wind data is presented and parameter estimates for the data are derived using the numerical approximation to the maximum likelihood presented in Chapter II.

II. MODELS WITH GAMMA MARGINALS

A. INTRODUCTION

There have been several schemes suggested for the modeling of dependent random variables with Gamma marginals. The Gamma autoregressive process of order one (GAR(1)) by Gaver and Lewis [Ref. 2], the discrete autoregressive process of order one (GDAR(1)) by Jacobs and Lewis [Ref. 12], the Gamma Beta autoregressive process of order one (GBAR(1)) by Fishman [Ref. 13] and Lawrance and Lewis [Ref. 14], and the Gamma autoregressive process of order one (GLAR(1)) by Lewis [Ref. 10]. There is also an attempt to use multivariate Gammas obtained by the inverse probability integral transform in a time series context by Schmeiser [Ref. 15].

The GAR(1) model generates an $\{X_n\}$ series using the standard first-order autoregression equation (first-order stochastic difference equation)

$$X_n = \rho X_{n-1} + \varepsilon_n, \quad 0 \le \rho < 1. \qquad \text{(II.A.1)}$$

The innovative factor, ε_n, has Laplace-Stieltjes transform of $(\rho+(1-\rho)\frac{\lambda}{\lambda+s})^k$ and the $\{X_n\}$ random variables have marginal Gamma distributions with shape parameter k and scale parameter λ. The marginal density function of the $\{X_n\}$ random variable is

$$f_{X_n}(x;k,\lambda) = f_X(x;k,\lambda) = \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda x}, \qquad \text{(II.A.2)}$$

$$\lambda > 0,\ x > 0,\ k > 0.$$

The model tends to produce runs of decreasing value when innovative term has successive realizations of value zero. The GAR(1) is in this sense highly degenerate, even though it is a true linear process. Ad hoc estimates of model parameters are available which produce the exact ρ value if the series is long enough [Ref. 2]. However, maximum likelihood estimates have not been produced. This model is not extendable to a moving average process.

The GDAR(1) produces an $\{X_n\}$ sequence using the first-order autoregressive equation with random coefficients.

$$X_n = V_n X_{n-1} + (1-V_n) G_n, \qquad \text{(II.A.3)}$$

where $\{V_n,\ n = 1,2,\ldots\}$ is an iid sequence of binary random variables with $P(V_n = 1) = 1-P(V_n = 0) = \rho$, $\{G_n,\ n = 1,2,\ldots\}$ is an iid Gamma sequence.

This sequence produces runs of constant value when successive realizations for V_n produce value 1. When V_n equals zero, a new value is selected. Obviously, this model has limited value in general applications and is even more degenerate than GAR(1) process.

The GBAR(1) is the most flexible model in that it contains the GAR(1) and GLAR(1) models as special cases. It produces

an $\{X_n\}$ sequence using

$$X_n = \beta B_n X_{n+1} + \varepsilon_n, \qquad \text{(II.A.4)}$$

where $\{B_n, n = 1,2,...\}$ is an iid Beta(k-q,q) sequence. ε_n was shown by Lawrance and Lewis to be the sum of a Gamma variable and the innovation process of the GAR(1) process [Ref. 14]. Although flexible in the sense that it contains the other models, it can not be extended to a moving average process. In addition, conditional densities and, hence, maximum likelihood estimates are not available. This is because the innovation random variable for the GAR(1) process, while it can be generated as a random sum of random variables, does not have a known distribution function.

The most valuable and flexible model seems to be the GLAR(1) which produces an $\{X_n\}$ sequence using the stochastic difference equation with random coefficients

$$X_n = B_n X_{n-1} + C_n G_n, \qquad \text{(II.A.5)}$$

where $\{X_n, n = 0,1,...\}$ is a second-order stationary sequence of Gamma random variables, $\{B_n, n = 1,2,...\}$ and $\{C_n, n = 1,2,...\}$ are iid Beta random variables, and $\{G_n\}$ is an iid sequence of Gamma random variables. This model is extendable as an autoregressive process of arbitrary order and as a moving average process of arbitrary order. These two forms can also be combined

to form a mixed model, the so-called Gamma Lewis autoregressive-moving average process of orders p and q (GLARMA(p,q)).

This chapter of the thesis examines some of the special cases of the model. One case in particular, the AR(1) form, is reasonably extensively examined. The correlation structure is developed. The conditional expectation and density are derived. The latter is used as the basis of a numerical approximation to the maximum likelihood method of parameter estimation. Directional moments and the probability of X_{n+1} being greater than X_n are derived. In a later Chapter of this thesis, this model is used as a basis for analyzing wind speed data.

The special case of the moving average of order one is examined in some detail. The correlation structure is derived with some emphasis on exploring the restrictions on the range of correlations that are possible. Directional moments and an empirical examination of the probability that X_{n+1} is greater than X_n are examined.

As a demonstration of the flexibility and extendability of the model the mixed model of order one, the autoregressive model of order two, and a bivariate model are introduced and their correlation structures derived.

B. FIRST-ORDER AUTOREGRESSIVE BETA-GAMMA MODEL, GLAR(1)

1. Introduction

The first-order autoregressive Beta-Gamma model is a special case of the GLARMA(p,q) model when q = 0 and p = 1.

The autoregressive model generates an $\{X_n\}$ sequence using

$$X_n = A_n X_{n-1} + B_n G_n, \tag{II.B.1.1}$$

where $\{X_n, n = 0,1,\dots\}$ is a second-order stationary sequence of random variables with a Gamma(k,1) marginal distribution; $\{A_n, n = 1,2,\dots\}$ is an iid sequence of Beta(k-q,q) random variables; $\{B_n, n = 1,2,\dots\}$ is an iid sequence of Beta(q,k-q) random variables; $\{G_n, n = 0,1,\dots\}$ is an iid sequence of Gamma (k,1) random variables; $\{A_n\}$, $\{B_n\}$, and $\{G_n\}$ are independent; $0 < q \leq k$.

Choosing $X_0 = G_0$ makes the $\{X_n\}$ sequence stationary. The Gamma random variables are parameterized by the shape parameter and the mean, rather than the scale parameter. This somewhat unusual parameterization has some advantages in statistical work since $\text{Gamma}(k,\mu) = \mu\,\text{Gamma}(k,1)$ [Ref. 7]. A Gamma(k,1) random variable has density

$$f_G(x;k,1) = \frac{k^k}{\Gamma(k)} x^{k-1} e^{-kx}, \quad x > 0,\ k > 0 \tag{II.B.1.2}$$

This is a special case of the more general density

$$f_G(x;k,\mu) = \frac{(\frac{k}{\mu})^k}{\Gamma(k)} x^{k-1} e^{-\frac{kx}{\mu}}$$

where $\mu = 1$. Of course, since the scale parameter, shape parameter and mean are related by the relation $\mu = \frac{k}{\lambda}$, the density can be specified by any two of the three parameters. The

typical parameterization in terms of the scale parameter, λ, is useful because $G(k_1,\lambda) + G(k_2,\lambda) = G(k_1+k_2,\lambda)$. This relationship is not true when the parameterization is through the shape parameter, k, and the mean, μ.

A Beta(m,n) random variable has density

$$f_B(x;m,n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1} (1-x)^{n-1}, \qquad \text{(II.B.1.3)}$$

$$0 < x < 1, \quad m > 0, \quad n > 0.$$

For the Beta random variable to be properly defined each of the parameters must be positive. Hence, when q = k, (II.B.1.1), as defined above, is no longer appropriate since each Beta random variable has a parameter that is identically zero. In this case when q = k it is understood that the $\{A_n\}$ sequence is considered to be identically zero and the $\{B_n\}$ sequence one. Therefore, II.B.1.1. becomes simply $X_n = G_n$, and the $\{X_n\}$ sequence, like the $\{G_n\}$ sequence, is iid. A justification of this generation scheme for a Gamma process as defined by II.B.1.1 was provided by Lawrance and Lewis [Ref. 14, pp.24].

In this section the correlation structures of the $\{X_n\}$ sequence and that of the $\{X_n\}$ and $\{G_n\}$ sequences are addressed. Other characteristics of the sequence, such as conditional expectation of X_n given X_{n-1}, directional moments, and $P(X_{n+1} > X_n)$ are considered. Of particular note is the derivation of the conditional density of X_n given X_{n-1}. This

leads to the formulation of the likelihood function and a computer program to generate maximum likelihood estimates of parameter values. The numerical convergence properties of the likelihood method are assessed in Section II.E.

2. Correlation Structure

The serial correlation of the $\{X_n\}$ series is easily determined by a straightforward calculation. We have

$$X_n = A_n X_{n-1} + B_n G_n$$

$$X_n X_{n-1} = A_n X_{n-1}^2 + B_n G_n X_{n-1}$$

Now X_i and G_j are independent if $j > i$ and X_i and A_j are independent if $j > i$. Using these facts along with the iid nature and independence of the $\{B_n\}$ and $\{G_n\}$ sequences yields the following expression when expectations are taken.

$$\begin{aligned} E(X_n X_{n-1}) &= E(A_n)E(X_{n-1}^2) + E(B_n)E(G_n)E(X_{n-1}) \\ &= \left(\frac{k-q}{k}\right) \cdot \frac{k(k+1)}{k^2} + \left(\frac{q}{k}\right) \cdot 1 \cdot 1 \\ &= \frac{k^2+k-kq-q+qk}{k^2} = \frac{k^2+k-q}{k^2} \end{aligned}$$

Therefore,

$$\mathrm{COV}(X_n, X_{n-1}) = \frac{k-q}{k^2}$$

and

$$\mathrm{CORR}(X_n, X_{n-1}) = \frac{k-q}{k} = 1 - \frac{q}{k}, \quad 0 < q \leq k. \qquad \text{(II.B.2.1)}$$

This is consistent with the fact that for $q = k$, $\{X_n\}$ is a sequence of i.i.d. Gamma variates which implies that $\mathrm{CORR}(X_n X_{n-1}) = 0$. This correlation is easly extended by an induction argument to yield

$$\mathrm{CORR}(X_n, X_{n+m}) = \left(\frac{k-q}{k}\right)^m, \quad n \geq m \geq 0. \qquad \text{(II.B.2.2)}$$

The two sequences $\{X_n\}$ and $\{G_n\}$ can be viewed as a bivariate pair (X_n, G_n) of Gamma(k,1) random variables. Therefore, the correlation structure of these sequences may be of interest. Proceeding as before

$$X_n = A_n X_{n-1} + B_n G_n$$

$$X_n G_n = A_n X_{n-1} G_n + B_n G_n^2$$

Taking expectations as before

$$E(X_n G_n) = E(A_n) E(X_{n-1}) E(G_n) + E(B_n) E(G_n^2)$$

$$E(X_n G_n) = \frac{k-q}{k} \cdot 1 \cdot 1 + \frac{q}{k} \cdot \frac{k(k+1)}{k^2} = \frac{k-q}{k} + \frac{kq+q}{k^2}$$

$$E(X_n G_n) = \frac{k^2+q}{k^2} \qquad \text{(II.B.2.3)}$$

Therefore

$$\mathrm{COV}(X_n, G_n) = \frac{q}{k^2}$$

and

$$\mathrm{CORR}(X_n, G_n) = \frac{q}{k} . \qquad \text{(II.B.2.4)}$$

When $q = k$ this is 1 since $X_n = G_n$.

Pursuing the process one more step we determine $\mathrm{CORR}(X_n, G_{n-1})$.

$$X_n = A_n X_{n-1} + B_n G_n,$$

$$X_n G_{n-1} = A_n X_{n-1} G_{n-1} + B_n G_n G_{n-1}$$

Taking expectations as before and using II.B.2.3 and the second-order stationarity of the $\{X_n\}$ sequence, we get

$$\begin{aligned} E(X_n G_{n-1}) &= E(A_n)E(X_{n-1}G_{n-1}) + E(B_n)E(G_n)E(G_{n-1}) \\ &= \left(\frac{k-q}{k}\right)\left(\frac{k^2+q}{k^2}\right) + \frac{q}{k}\cdot 1\cdot 1 = \frac{k^3+kq-k^2q-q^2+qk^2}{k^3} \\ &= \frac{k^3+kq-q^2}{k^3} \end{aligned}$$

Therefore,

$$\mathrm{COV}(X_n, G_{n-1}) = \frac{q(k-q)}{k^3} \qquad \text{(II.B.2.5)}$$

and

$$\mathrm{CORR}(X_n, G_{n-1}) = \left(\frac{q}{k}\right)\left(\frac{k-q}{k}\right) \qquad \text{(II.B.2.6)}$$

II.B.2.3 through II.B.2.6 can be used in a simple induction argument to yield the general result

$$\mathrm{CORR}(X_n, G_{n-m}) = \left(\frac{q}{k}\right)\left(\frac{k-q}{k}\right)^m, \quad m = 0,1,\ldots,n. \qquad \text{(II.B.2.7)}$$

When j is greater than i, X_i and G_j are independent. Hence, $\mathrm{CORR}(X_i, G_j) = 0$, $j > i$.

3. Conditional Expectation and Conditional Density

The conditional expectation of X_n given $X_{n-1} = y$ is trivially determined from the defining equation and the moments of the Beta distribution as

$$E(X_n | X_{n-1} = y) = (\frac{k-q}{k})y + \frac{q}{k}. \qquad \text{(II.B.3.1)}$$

Recognizing $\frac{k-q}{k}$ as the correlation between X_n and X_{n-1} and letting ρ be that correlation, II.B.3.1 can be written as

$$E(X_n | X_{n-1} = y) = \rho y + \frac{q}{k} = \rho y + (1-\rho). \qquad \text{(II.G.3.2)}$$

Thus the regression is linear in y.

The conditional density of X_n given X_{n-1} can also be determined. It is easiest to start by deriving the conditional distribution of X_n given $X_{n-1} = y$.

$$P(X_n \leq x | X_{n-1} = y) = P(A_n y + B_n G_n \leq x) = P(B_n G_n \leq x - A_n y)$$

Now [Ref. 14] the product of a Beta(q,k-q) random variable and a Gamma(k,1) random variable is a random variable with density Gamma($q,\frac{q}{k}$). Hence, if we let D_n be a Gamma($q,\frac{q}{k}$) random variable,

$$P(X_n \leq x | X_{n-1} = y) = P(D_n \leq x - A_n y) \qquad \text{(II.B.3.3)}$$

This can be written as a convolution if one is careful about the upper limit of integration. Since D_n is Gamma($q,\frac{q}{k}$), it is non-negative. Hence, $P(D_n \leq x-A_n y)$ is zero if $x-A_n y$ is

less than zero. Since A_n is a Beta random variable and, hence, bounded above by one, $x-A_n y$ can not be negative if $x > y$. However, if $y > x$, then A_n must be restricted to lie in the range $(0,\frac{x}{y})$. Taking this restriction into account and writing the RHS of II.B.3.3 as a convolution

$$P(X_n \le x | X_{n-1} = y) = \int_0^L f_A(z) F_D(x-yz)\, dz, \qquad \text{(II.B.3.4)}$$

and

$$L = \begin{cases} 1 & \text{if } x \ge y, \\ \frac{x}{y} & \text{if } x < y, \end{cases}$$

where $f_A(z)$ is the density of a Beta random variable as in II.B.1.3, and F_D is the distribution function of a Gamma $(q,\frac{q}{k})$ random variable.

Of course, to get the conditional density of X_n given $X_{n-1} = y$, we must take the derivative of II.B.3.4 with respect to x. Recognizing that the upper limit may be a function of x and applying Leibneitz's Rule where appropriate yields

$$f_{X_n|X_{n-1}}(x) = \int_0^L f_A(z) f_D(x-yz)\, dy, \qquad \text{(II.B.3.5)}$$

and

$$L = \begin{cases} 1 & \text{if } x \ge y, \\ \frac{x}{y} & \text{if } x < y, \end{cases}$$

where $f_{X_n|X_{n-1}}(x)$ is the conditional density of X_n given X_{n-1}, $f_A(z)$ is the Beta density, and $f_D(x-yz)$ is the Gamma $(q,\frac{q}{k})$ density as in II.B.1.2. Writing this result in terms of the densities involved we have

$$f_{X_n|X_{n-1}}(x) = \int_0^L \frac{\Gamma(k)}{\Gamma(x-q)\Gamma(q)} z^{k-q-1}(1-z)^{q-1}\frac{k^q}{\Gamma(q)}(x-yz)^{q-1}e^{-k(x-yz)}dy,$$

with the condition on L, as in II.B.3.5. And finally,

$$f_{X_n|X_{n-1}}(x) = \left\{\frac{\Gamma(k)}{\Gamma(k-q)\,[\Gamma(q)]^2}\right\}k^q e^{-kx}$$

$$\times \int_0^L z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dy, \qquad \text{(II.B.3.6)}$$

and

$$L = \begin{cases} 1 & \text{if} \quad x \geq y, \\ \frac{x}{y} & \text{if} \quad x < y. \end{cases}$$

As a check on the derivation of the conditional density, the conditional density and conditional expectation were calculated for values of k and q which produced simple integrands. One of these cases was $k = 2$, $q = 1$. Then $k-q-1 = 0$ and $q-1 = 0$. In these parameter values II.B.3.6 reduces to

$$f_{X_n|X_{n-1}}(x) = 2e^{-2x}\int_0^L e^{2yz}dz \qquad L = \begin{cases} 1 & \text{if} \quad x \geq y, \\ \frac{x}{y} & \text{if} \quad x < y. \end{cases}$$

After integration,

$$f_{X_n|X_{n-1}}(x) = \begin{cases} e^{-2x}[\frac{e^{2y}}{y} - \frac{1}{y}] & \text{if} \quad x \geq y, \\ \frac{1}{y} - \frac{e^{-2x}}{y} & \text{if} \quad x < y. \end{cases} \qquad \text{(II.B.3.7)}$$

Since the two expressions in II.B.3.7 are non-negative, we can insure that this is a density by verifying that its integral is one.

$$\int_0^\infty f_{X_n|X_{n-1}}(x)\,dx = \int_0^y [\frac{1}{y} - \frac{e^{-2x}}{y}]\,dx + \int_y^\infty e^{-2x}[\frac{e^{2y}}{y} - \frac{1}{y}]\,dx$$

$$= \frac{1}{y}\int_0^y dx - \frac{1}{y}\int_0^y e^{-2x}dx + [\frac{e^{2y}}{y} - \frac{1}{y}]\int_y^\infty e^{-2x}dx$$

$$= 1 + \frac{e^{-2y}}{2y} - \frac{1}{2y} + \frac{1}{2y} - \frac{e^{-2y}}{y}$$

$$= 1,$$

as required. We can also take the conditional expectation to see if it equals $\frac{y}{2} + \frac{1}{2}$ as required by II.B.3.1. Thus,

$$\int_0^\infty x f_{X_n|X_{n-1}}(x)dx = \int_0^y x[\frac{1}{y} - \frac{e^{-2x}}{y}]\,dx + \int_y^\infty xe^{-2x}[\frac{e^{2y}}{y} - \frac{1}{y}]\,dx$$

$$\int_0^\infty x f_{X_n|X_{n-1}}(x)dx = \frac{1}{y}\int_0^y x dx - \frac{1}{y}\int_0^y xe^{-2x}dx + [\frac{e^{2y}}{y} - \frac{1}{y}]\int_y^\infty xe^{-2x}dx$$

$$= \frac{y}{2} + \frac{e^{-2y}}{2} + \frac{e^{-2y}}{4y} - \frac{1}{4y} + \frac{1}{2} + \frac{1}{4y} - \frac{e^{-2y}}{2} - \frac{e^{-2y}}{4y}$$

$$= \frac{y}{2} + \frac{1}{2},$$

as required.

Using k = 3 and q = 1 produces a density of

$$f_{X_n|X_{n-1}}(x) = \begin{cases} 2e^{-3x}[\frac{1}{y}(e^{3y} - \frac{e^{3y}}{3y} + \frac{1}{3y})] & \text{if } x \geq y, \\ \frac{2x}{y^2} - \frac{2}{3y^2}(1 - e^{-3x}) & \text{if } x < y. \end{cases} \quad \text{(II.B.3.8)}$$

This density is non-negative and integrates to one. It also produces a conditional expectation of $\frac{2y}{3} + \frac{1}{3}$ as required by II.B.3.1. These results are also of use in validating the results obtained from numerical integrations of II.B.3.6 in estimation applications.

This conditional density can be used to form the joint density of $X_1, X_2, \ldots, X_n$ and, hence, the likelihood function of the data. This subject will be addressed in the following section.

4. Maximum Likelihood Estimation

Once the conditional density of X_n given X_{n-1} and the marginal density of X_1 are known, it is possible to evaluate

the joint density of $X_1, X_2, \ldots, X_n$. Since the first-order autoregressive process is Markovian, as can be seen directly from the defining equation II.B.1.1, the following equation is valid:

$$f(x_j | x_{j-1}, x_{j-2}, \ldots, x_i) = f(x_j | x_{j-1}), \quad n \geq j \geq 2 \qquad \text{(II.B.4.1)}$$

Applying II.B.4.1 n-1 times to the joint density of $X_n, X_{n-1}, \ldots, X_1$ produces

$$f(x_n, x_{n-1}, \ldots, x) = f_1(x_n | x_{n-1}) f_1(x_{n-1} | x_{n-2}) \ldots f_1(x_2 | x) f_2(x_1) \qquad \text{(II.B.4.2)}$$

where f is the joint density of $X_n, \ldots, X_1$; f_1 is the conditional density of X_j given X_{j-1}; and f_2 is the marginal density of the $\{X_n\}$ sequence.

Viewing the joint density as the likelihood function, letting L be the likelihood function, and taking natural logarithms of each side of II.B.4.2 produces

$$\ln L = \ln f_2(x_1) + \sum_{i=1}^{n-1} \ln f_1(x_{i+1} | x_i) \qquad \text{(II.B.4.3)}$$

Recall that in Section II.B.3 the conditional density was determined to be

$$f_1(x|y) = \left\{\frac{\Gamma(k)}{\Gamma(k-q)[\Gamma(q)]^2}\right\} k^q e^{-kx} \int_0^L z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1} e^{kyz} dz,$$

$$L = \begin{cases} 1 & \text{if } x \geq y, \\ \frac{x}{y} & \text{if } x < y. \end{cases} \qquad \text{(II.B.4.4)}$$

For a given set of data the likelihood can be viewed as a function of the parameters k and q. Let

$$G(k,q) = \ln f_2(x_1) + \sum_{i=1}^{n-1} \ln f_1(x_{i+1}|x_i) \qquad \text{(II.B.4.5)}$$

Assume for the moment that a procedure has been established to evaluate G(k,q) given k,q ($0 < q \leq k$) and the data. Consider the problem of constructing a program to determine the values of k and q which maximize G(k,q). An outline of the program can be constructed as follows:

1. Read the initial values for k and q.
   Read the data.
2. Determine a direction of search. Use the following difference equations to approximate derivatives.

$$\frac{\partial}{\partial k} G(k,q) \approx \frac{G(k+\Delta k,q) - G(k,q)}{\Delta k} \qquad \text{(II.B.4.6)}$$

$$\frac{\partial}{\partial q} G(k,q) \approx \frac{G(k,q+\Delta q) - G(k,q)}{\Delta q} \qquad \text{(II.B.4.7)}$$

Let $Dk_i = \frac{G(k+\Delta k,q) - G(k,q)}{\Delta k}$ and $Dq_i = \frac{G(k,q+\Delta q) - G(k,q)}{\Delta q}$ for the ith iteration. If we define $\nabla G_i = \binom{Dk_i}{Dq_i}$, then ∇G_i approximates the gradient of G at the current k,q values. For i = 1, let the direction of search, d_1, be ∇G_i. For i > 1, define the direction of search, d_k, to be

$$d_k = \nabla G_i + \beta_{k-1} d_{k-1}, \qquad \text{(II.B.4.8)}$$

where β_{k-1} is $\frac{\nabla G_i^T \nabla G_i}{\nabla G_{i-1}^T \nabla G_{i-1}}$, the ratio of the length of the present and preceding gradients. Formula II.B.4.8 is the key equation in implementing the Fletcher-Reeves Conjugate Gradient Algorithm. Once d_k has been selected, normalize its length to one.

3. Let the initial step length, SL, be 10^{-3} and let N = 1. Compute the trial values of k and q, Tk and Tq, using

$$\begin{aligned} Tk &= k + SL * Dk_i, \\ Tq &= a + SL * Dq_i. \end{aligned} \qquad \text{(II.B.4.9)}$$

If G(Tk,Tq) > G(k,q), let SL = 2 * SL, otherwise go to 4. If N = 10; k = Tk, q = Tq, go to 2. (No step larger than $2^{10} * 10^{-3} \approx 1.0$ is allowed.) If n < 10; N = N+1, go to 3.

4. If N > 1 (at least one step produced an increase), use a golden section search between Tk,Tq for step N-2 and

Tk and Tq for step N to determine the maximum function value and the k and q values, kMAX and qMAX, which produced it. Here k = kMAX, q = qMAX, go to 2.

If N = 1, go to 5.

5. Since the initial step along the indicated direction produced a decrease in the function value, check to see if you are at a local maximum. Determine the function value at points at 30° intervals on the circumference of circles with radii of 10^{-3}, 10^{-2}, 10^{-1} (0° is parallel to the q axis). If the maximum of these test values is greater than the present value, set k and q to the values which produced the maximum value, set the present function value to the maximum, set i = 1 and go to 2. Otherwise terminate.

The program above assumed that, given q, k and the data, the value of the likelihood could be calculated. One difficulty in performing the calculation is that the integrand of the conditional density may contain singularities. As a precondition to using an IMSL routine to evaluate the integral, these potential singularities must be removed. The technique employed requires that the coefficient of the term that goes to infinity as one of the limits of integration is approached is added and subtracted. The part that is added is then integrated separately and added to the part that is evaluated by the IMSL routine.

To procede with this technique we first split the integral into two parts. Thus, ignoring the part of II.B.4.4

outside the integral, we have

$$\int_0^L z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz = \int_0^{L/2} z^{k-q-1}(1-z)^{q-1}$$

$$\times (x-yz)^{q-1}e^{kyz}dz + \int_{L/2}^{L} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz \tag{II.B.4.10}$$

In the first part of the RHS of II.B.4.10 the term z^{k-q-1} could tend to infinity as z tends to zero if $k-q-1 < 0$. If we set z equal to zero in the remainder of the integrand we get $(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}\big|_{z=0} = x^{q-1}$. Adding and subtacting this term times the term that contains the singularity, we have

$$\int_0^{L/2} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz$$

$$= \int_0^{L/2} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz} - x^{q-1}z^{k-q-1} + x^{q-1}z^{k-q-1}dz$$

$$= \int_0^{L/2} z^{k-q-1}[(1-z)^{q-1}(x-yz)^{q-1}e^{kyz} - x^{q-1}]dz + x^{q-1}\int_0^{L/2} z^{k-q-1}dz$$

Thus,

$$\int_0^{L/2} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz$$

$$= \int_0^{L/2} z^{k-q-1}[(1-z)^{q-1}(x-yz)^{q-1}e^{kyz} - x^{q-1}]dz + x^{q-1}\frac{(\frac{L}{2})^{k-q}}{k-q}. \tag{II.B.4.11}$$

Recall that since $q < k$, $k-q > 0$ and the second integral on the RHS can, in fact, be integrated as shown. Now the integral on the RHS can be evaluated by IMSL routine DCADRE and the second part of the RHS can be easily computed.

Applying this technique to the second half of the integral, recalling that the case where $L = 1$ and $L = \frac{x}{y}$ must be considered separately produces

$$\int_{1/2}^{1} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz$$

$$= \int_{1/2}^{1} [z^{k-q-1}(x-yz)^{q-1}e^{kyz} - (x-y)^{q-1}e^{ky}]\,(1-z)^{q-1}dz \qquad \text{(II.B.4.12)}$$

$$+ (x-y)^{q-1}e^{ky}\,\frac{(1/2)^q}{q}$$

and

$$\int_{x/2y}^{x/y} z^{k-q-1}(1-z)^{q-1}(x-yz)^{q-1}e^{kyz}dz$$

$$= \int_{x/2y}^{x/y} (x-yz)^{q-1}[z^{k-q-1}(1-z)^{q-1}e^{kyz} - (\tfrac{x}{y})^{k-q-1}(1-\tfrac{x}{y})^{q-1}e^{kx}]\,dz$$

$$+ (\tfrac{x}{y})^{k-q-1}(1-\tfrac{x}{y})^{q-1}e^{kx}\,\frac{(\frac{x}{2})^q}{yq}. \qquad \text{(II.B.4.13)}$$

Since $q > 0$, the second terms in the two previous expressions are properly integrated.

Two points should be noted. First, if $k-q-1 > 0$ or if $q-1 > 0$, then these steps are not required. But whether they are required or not, they are always accurate. Since the exact path of the search algorithm is unknown at the start, these expressions are used throughout to insure accurate calculations regardless of the k and q values encountered. Second, if $x = y$, two parts of the integrand simultaneously tend to infinity and this procedure breaks down. This does not pose a problem for continuous data since the probability of this occurring is zero. However, if discrete values are used or if the data is truncated because of limits on the accuracy of the measuring device, then the data may have to be preprocessed to insure that successive values are not equal by adding a small increment to one of the values.

When the program was written, its accuracy was verified by three checkes. The case $k = q$ implies independence in the basic model since as q tends to k the probability that A_n equals zero tends to one and the probability that B_n equals one tends to one. Hence, in the limit $X_n = G_n$ and G_n is an iid sequence. The logarithm of the likelihood function for independence was calculated and compared to the program results for several values of k and q where $k = q$. The two calculations were equal within machine roundoff and computational accuracy. The special cases of $k = 2$, $q = 1$ and $k = 3$, $q = 1$ discussed in II.B.3 were also computed. The logarithm of the likelihood function was computed using each

of the conditional densities derived in II.B.3.7 and II.B.3.8. When the results of these calculations were compared to the program results with the specified k and q values, the results were equal within calculation and roundoff accuracy.

The results of the tests of this program when used with simulated data with known parameter values are presented in Section II.E.

Note that there are natural moment estimators for the three parameters, k and q and μ in this model. These follow from the fact that

$$\rho(1) = \mathrm{CORR}(X_n X_{n+1}) = 1 - \frac{q}{k};$$

$$C^2(X) = \frac{\mathrm{Var}(X_n)}{[E(X_n)]^2} = \frac{\mathrm{Var}(X)}{[E(X)]^2} = \frac{\sigma_x^2}{\mu^2} = \frac{1}{k}$$

Thus we use for moment estimations

$$\hat{\mu} = \bar{x} \qquad \text{(II.B.4.14)}$$

$$\hat{q} = (1-\hat{\rho}(1)\hat{k} \qquad \text{(II.B.4.15)}$$

$$\hat{k} = \frac{(\bar{x})^2}{s_x^2} \qquad \text{(II.B.4.16)}$$

These moment estimations will be compared to maximum likelihood estimations in the case where μ = 1 in Section II.E.

5. Directional Moments

Unlike processes with normal marginals, non-normal processes are not completely determined by their correlation structure. Directional moments not only demonstrate this difference, but also help to differentiate processes with similar correlation structure and identical marginal distributions. They can also be used to help determine parameter values. In addition, they may also be viewed as another way of characterizing the joint distribution of the process. With

$$X_n = A_n X_{n-1} + B_n G_n,$$

with all random variables defined as in II.B.1.1, we first address $E(X_n X_{n-1}^2)$. We have

$$X_n = A_n X_{n-1} + B_n G_n$$

$$X_n X_{n-1}^2 = A_n X_{n-1}^3 + B_n G_n X_{n-1}^2.$$

Taking expectations, recalling G_i and X_j are independent if $i > j$, $\{B_n\}$ and $\{G_n\}$ are independent, and A_i and X_j are independent if $i > j$. Then

$$E(X_n X_{n-1}^2) = E(A_n)E(X_{n-1}^3) + E(B_n)E(G_n)E(X_{n-1}^2)$$

$$= \frac{k-q}{k} \frac{k(k+1)(k+2)}{k^3} + \frac{q}{k} \cdot 1 \cdot \frac{k(k+1)}{k^2}$$

$$E(X_n X_{n-1}^2) = \frac{k(k+1)}{k^3}\left[\frac{(k-q)}{k} \cdot (k+2) + q\right]$$

$$E(X_n X_{n-1}^2) = \frac{k(k+1)}{k^3}\left[\frac{k^2+2(k-q)}{k}\right] \qquad \text{(II.B.5.1)}$$

Since

$$X_n^2 = A_n^2 X_{n-1}^2 + 2A_n B_n G_n X_{n-1} + B_n^2 G_n^2,$$

$$X_n^2 X_{n-1} = A_n^2 X_{n-1}^3 + 2A_n B_n G_n X_{n-1}^2 + B_n^2 G_n^2 X_{n-1}$$

Taking expectations as before

$$\begin{aligned} E(X_n^2 X_{n-1}) &= E(A_n^2)E(X_{n-1}^3) + 2E(A_n)E(B_n)E(G_n)E(X_{n-1}^2) \\ &\quad + E(B_n^2)E(G_n^2)E(X_{n-1}) \\ &= \frac{(k-q)(k-q+1)}{k(k+1)} \frac{k(k+1)(k+2)}{k^3} + \frac{2(k-q)}{k} \cdot \frac{q}{k} \cdot 1 \cdot \frac{k(k+1)}{k^2} \\ &\quad + \frac{q(q+1)}{k(k-1)} \frac{k(k+1)}{k^2} \cdot 1 \end{aligned}$$

After simplification we get

$$E(X_n^2 X_{n-1}) = \frac{(k-q)(k+1)(k+2)}{k^3} + \frac{qk(k+1)}{k^3} \qquad \text{(II.B.5.2)}$$

Note that these two directional moments are different, indicating that the process is not time reversible.

6. $P(X_{n+1} > X_n)$

Another characterization of the joint distribution and the time directionality of the process is given by $P(X_{n+1} > X_n)$. There is a simple analytical solution for $P(X_{n+1} > X_n)$ in the GLAR(1) process. Consider,

$$P(X_{n+1} > X_n) = P(A_{n+1}X_n + B_{n+1}G_{n+1} > X_n)$$

$$= P(B_{n+1}G_{n+1} > [1-A_{n+1}]X_n) \qquad \text{(II.B.6.1)}$$

Recall that B_{n+1} is Beta(q,k-q) and A_{n+1} is Beta(k-q,q). Hence, $[1-A_{n+1}]$ is Beta(q,k-q). Since G_{n+1} and X_n are independent and have the same marginal distribution and A_{n+1} and B_{n+1} are independent, each side of the inequality in II.B.6.1 has the same distribution and the random variables are independent. Hence

$$P(X_{n+1} > X_n) = 0.50.$$

This is a strong property of the process. While the process, as seen by the directional moments, is not time reversible, the fact that $P(X_{n+1} > X_n) = 0.50$ means that sample paths will have as many "runs up" as "runs down". Sample paths are given in Figures II.B.6.1 and II.B.6.2.

An additional point of interest occurs when k = 1 and the process has exponential marginal distributions.

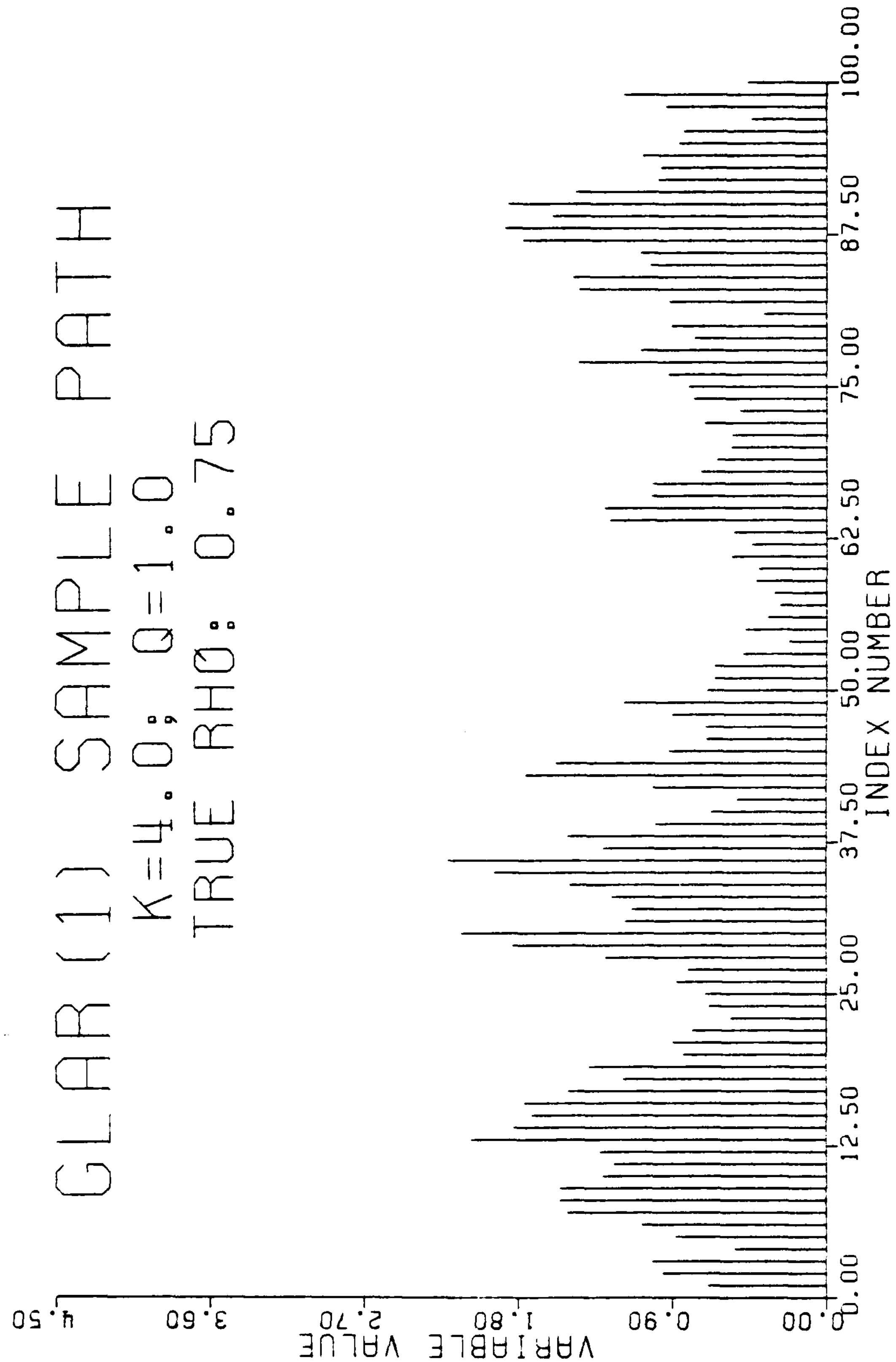


Figure II.B.6.1

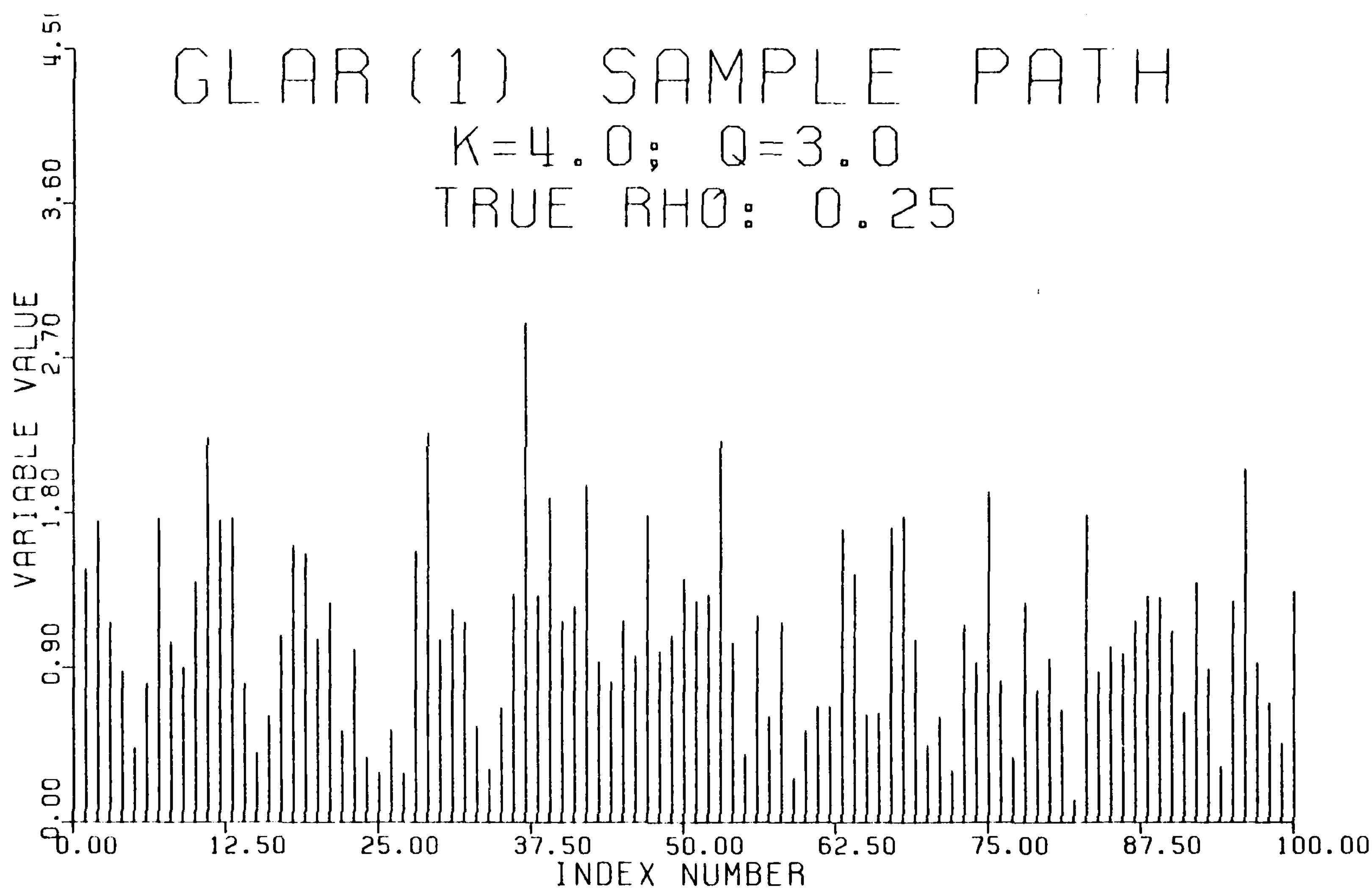


Figure II.B.6.2

Another exponential first-order autoregressive process in which $P(X_{n+1} > X_n) = 0.50$ is the PREAR(1) process. This is a special case of the two parameter NEAR(1) exponential process of Lawrance and Lewis [Ref. 8] in which the two parameters α and β are related by $\beta = \frac{1}{2-\alpha}$. The two exponential processes are very similar in sample path properties. However, the GLAR(1) process has a smoother joint distribution. In fact, the likelihood for the PREAR(1) process is discontinuous, making it difficult to get maximum likelihood estimators.

C. FIRST-ORDER MOVING AVERAGE BETA-GAMMA MODEL, GLMA(1)

1. Introduction

Another special case of the GLARMA(p,q) model is the first-order moving average model where $p = 0$ and $q = 1$. This arises naturally from the key result that an $\{X_n\}$ sequence can be formed by the random sum of two independent Gamma random variables. In the first-order autoregressive case of Section II.B, the generation scheme was given by equation II.B.1.1 and is repeated here

$$X_n = A_n X_{n-1} + B_n G_n.$$

The distribution of the $\{X_n\}$ sequence depends on the independence and distribution characteristics of X_{n-1} and G_n. It should be noted, however, that any two independent random variables with the required Gamma distribution could be

substituted for X_{n-1} and G_n without changing the distribution of X_n. In particular if we substitute G_n for X_{n-1} and G_{n-1} for G_n, we produce the first-order moving average process which generates the $\{X_n\}$ sequence using

$$X_n = A_n G_n + B_n G_{n-1}, \qquad \text{(II.C.1.1)}$$

where $\{X_n, n = 1,2,...\}$ is a second-order stationary sequence of random variables with marginal Gamma(k,1) distributions, $\{A_n, n = 1,2,...\}$ is an iid sequence of Beta(k-q,q) random variables; $\{B_n, n = 1,2,...\}$ is an iid sequence of Beta(q,k-q) random variables; $\{G_n, n = 0,1,...\}$ is an iid sequence of Gamma(k,1) random variables; $\{A_n\}$, $\{B_n\}$, and $\{G_n\}$ are independent; $0 < q \leq k$. The Gamma random variables are parameterized as in II.B.1 with density as in II.B.1.2. The Beta random variables have density as in II.B.1.3.

In this section we will address the correlation structure of the $\{X_n\}$ sequences and that of the $\{X_n\}$, $\{G_n\}$ sequences, theoretical ranges for possible correlations for the $\{X_n\}$ sequences, directional moments, and the $P(X_{n+1} > X_n)$.

2. Correlation Structure

Using II.C.1.1 to define X_n and X_{n-1} we have

$$X_n X_{n-1} = (A_n G_n + B_n G_{n-1})(A_{n-1} G_{n-1} + B_{n-1} G_{n-2})$$

$$= A_n A_{n-1} G_n G_{n-1} + A_n B_{n-1} G_n G_{n-2} + A_{n-1} B_n G_{n-1}^2 + B_n B_{n-1} G_{n-1} G_{n-2}.$$

Using the iid nature and independence of $\{A_n\}$, $\{B_n\}$, and $\{G_n\}$ and taking expectations

$$E(X_nX_{n-1}) = E(A_n)E(A_{n-1})E(G_n)E(G_{n-1})+E(A_n)E(B_{n-1})E(G_n)E(G_{n-2})$$

$$+ E(A_{n-1})E(B_n)E(G^2_{n-1})+E(B_n)E(B_{n-1})E(G_{n-1})E(G_{n-2})$$

$$= (\frac{k-q}{k})^2 + (\frac{k-q}{k})(\frac{q}{k}) + (\frac{k-q}{k})(\frac{q}{k})(\frac{k[k+1]}{k^2}) + (\frac{q}{k})^2$$

$$= 1 + (\frac{k-q}{k})(\frac{q}{k})(\frac{1}{k}) \qquad \text{(II.C.2.1)}$$

Therefore,

$$COV(X_n, X_{n-1}) = (\frac{k-q}{k})(\frac{q}{k})(\frac{1}{k})$$

and

$$CORR(X_n, X_{n-1}) = (1 - \frac{q}{k})(\frac{q}{k}) . \qquad \text{(II.C.2.2)}$$

A simple calculation will show that for lags greater than one the correlation is zero. So equation II.C.2.2 plus the knowledge that greater lags are zero is sufficient to specify the correlation structure of the $\{X_n\}$ sequence.

One might note at this juncture that the correlation of the $\{X_n\}$ sequence is constrained to lie in the interval $(0,\frac{1}{4})$. The reason for this constraint and a method for relaxing it will be discussed later. It is also worthy of

note that the $\{X_n\}$ sequence is stationary and has the same marginal distribution as the $\{G_n\}$ sequence, if $X_0 = G_0$.

As indicated in II.B.2 the $\{X_n\}$ and $\{G_n\}$ sequences can be considered to be a bivariate, correlated Gamma(k,1) process. As such, the correlation structure of this bivariate Gamma may be of interest. Consequently, we first develop the correlation of X_n and G_n in the standard fashion.

$$X_n = A_n G_n + B_n G_{n-1}$$

$$X_n G_n = A_n G_n^2 + B_n G_n G_{n-1}$$

Taking expectations as before,

$$\begin{aligned} E(X_n G_n) &= E(A_n)E(G_n^2) + E(B_n)E(G_n)E(G_{n-1}) \\ &= \left(\frac{k-q}{k}\right)\left(\frac{k[k+1]}{k^2}\right) + \frac{q}{k} \\ &= \frac{k^2 + k - q}{k^2} \end{aligned}$$

Therefore,

$$\mathrm{COV}(X_n, G_n) = \frac{k-q}{k^2}$$

and

$$\mathrm{CORR}(X_n, G_n) = \frac{k-q}{k} . \tag{II.C.2.3}$$

Now consider the correlation of X_n and G_{n-1}. We have

$$X_n = A_n G_n + B_n G_{n-1}$$

$$X_n G_{n-1} = A_n G_n G_{n-1} + B_n G_{n-1}^2 .$$

Expectations in the standard fashion produce

$$\begin{aligned} E(X_n G_{n-1}) &= E(A_n)E(G_n)E(G_{n-1}) + E(B_n)E(G_{n-1}^2) \\ &= \frac{k-q}{k} + \frac{q}{k}\left(\frac{k[k+1]}{k^2}\right) \\ &= \frac{k^2+q}{k^2} \end{aligned}$$

Hence,

$$\mathrm{COV}(X_n, G_{n-1}) = \frac{q}{k^2}$$

And finally

$$\mathrm{CORR}(X_n, G_{n-1}) = \frac{q}{k} . \tag{II.C.2.4}$$

A simple calculation convinces one that the correlation for lags greater than one is zero. In addition, it is clear that $\mathrm{CORR}(X_n, G_{n+m}) = 0$ for $m = 1,2,\ldots$ Hence, II.C.2.3 and

II.C.2.4 are sufficient to specify the correlation structure of the $\{X_n\}$ and $\{G_n\}$ sequences.

It has been noted before that the range of correlations for the first-order moving average process generated by the Beta-Gamma method of II.C.1.1 is constrained to lie in the interval from zero to one-quarter. There may be other random linear combinations of Gamma random variables which give a moving average process with Gamma marginals and a greater range of positive correlations. Thus we now examine a more general hypothetical generation process to prove that any random, linear combination of two independent Gamma random variables which generates a sequence with the same first two moments as those Gamma variables has a correlation that lies in this same interval. In fact, this proof only requires that the dependent random variable have the same first two marginal moments as the generating Gamma variables.

THEOREM:

If the $\{X_n\}$ sequence is generated by

$$X_n = A_n G_n + B_n G_{n-1}, \tag{II.C.2.5}$$

where $\{X_n\}$ is a second-order stationary, non-negative sequence of random variables with the same first two moments as the $\{G_n\}$ sequence; $\{A_n\}$ and $\{B_n\}$ are iid sequences of random variables; $\{G_n\}$ is an iid sequence of Gamma(k,1) random

variables; and $\{A_n\}$, $\{B_n\}$, and $\{G_n\}$ are independent, then $0 \leq CORR(X_n, X_{n-1}) \leq 0.25$.

<u>Proof</u>: Since $\{A_n\}$, $\{B_n\}$, and $\{G_n\}$ are independent and $\{X_n\}$ is non-negative, $\{A_n\}$ and $\{B_n\}$ must be non-negative. Hence $E(A) \geq 0$ and $E(B) \geq 0$.

Taking expectations of II.C.2.5, we have

$$E(X_n) = E(A_n)E(G_n) + E(B_n)E(G_{n-1})$$

$$1 = E(A) + E(B) \qquad \text{(II.C.2.6)}$$

Hence, $0 \leq E(A) \leq 1$ and $0 \leq E(B) \leq 1$.

Computing the serial correlation of $\{X_n\}$ yields

$$X_n X_{n-1} = (A_n G_n + B_n G_{n-1})(A_{n-1} G_{n-1} + B_{n-1} G_{n-2})$$

$$= A_n A_{n-1} G_n G_{n-1} + A_n B_{n-1} G_n G_{n-2} + A_{n-1} B_n G_{n-1}^2 + B_n B_{n-1} G_{n-1} G_{n-2}$$

Then

$$E(X_n X_{n-1}) = E(A_n)E(A_{n-1})E(G_n)E(G_{n-1}) + E(A_n)E(B_{n-1})E(G_n)E(G_{n-2})$$

$$+ E(A_{n-1})E(B_n)E(G_{n-1}^2) + E(B_n)E(B_{n-1})E(G_{n-1})E(G_{n-2})$$

$$= 1 + E(A)E(B)\left(\frac{1}{k}\right).$$

Since $E(X_n) = E(G_n) = 1$,

$$COV(X_n, X_{n-1}) = E(A)E(B)(\frac{1}{k})$$

And since $VAR(X_n) = VAR(G_n)$,

$$CORR(X_n, X_{n-1}) = E(A)E(B)$$

Using II.C.2.6 and its consequences, we have

$$0 \leq CORR(X_n, X_{n-1}) = E(A)[1 - E(A)] \leq \frac{1}{4}. \quad \text{(II.C.2.7)}$$

So, in general, if $\{X_n\}$ is second-order stationary with the same first two moments as $\{G_n\}$, the serial correlation of $\{X_n\}$ is bounded below by zero and above by one-fourth. Q.E.D.

This constraint on the correlation appears to be restrictive since, in the classical case, when two normally distributed random variables are added to produce a normal sequence, the range of correlations is $(-\frac{1}{2}, \frac{1}{2})$. The two situations, however, are not comparable. It is clear upon reflection that the constraints imposed on the $\{X_n\}$ sequence in the previous theorem are more severe than those imposed upon the classical normal case. In the above theorem we required that both the mean and variance of the $\{X_n\}$ sequence equal that of the innovative sequence. However, in the classical normal case (where zero mean normals are used as innovative factors)

only the mean of the generated sequence is equal to the mean of the innovative sequence. The variances are not equal. We now examine the case where the generated sequence is required to have the same mean as the innovative sequence, but is free to have a different variance. (This is the case with the usual constant coefficient, linear additive MA(1) scheme.)

THEOREM

If the non-negative $\{X_n\}$ sequence is generated by II.C.2.5 and all variables are defined as for that equation except that $\{X_n\}$ is only constrained to have the same first moment as $\{G_n\}$, then $0 \leq \mathrm{CORR}(X_n, X_{n-1}) \leq 0.5$.

Proof: Taking expectations of II.C.2.5 with the new circumstances produces

$$E(X) = [E(A) + E(B)]E(G)$$

$$1 = E(A) + E(B)$$

and $0 \leq E(A) \leq 1$, $0 \leq E(B) \leq 1$ by following reasoning identical to that above. Calculation to determine the serial correlation can initially proceed as usual.

$$X_n X_{n-1} = (A_n G_n + B_n G_{n-1})(A_{n-1} G_{n-1} + B_{n-1} G_{n-2})$$

$$E(X_n X_{n-1}) = 1 + E(A)E(B)(\tfrac{1}{k})$$

$$COV(X_n, X_{n-1}) = E(A)E(B)(\frac{1}{k}).$$

To this point all calculations and reasoning have been the same as that which produced II.C.2.7. However, since $\{X_n\}$ is not constrained to have the same second moment as $\{G_n\}$ the most explicit result obtainable is

$$CORR(X_n, X_{n-1}) = \frac{E(A)E(B)(\frac{1}{k})}{VAR(X)}, \qquad (II.C.2.8)$$

where VAR(X) is, of course, a function of VAR(A), VAR(B), and VAR(G). Since it is obvious that the smaller the value for VAR(X), the greater the serial correlation for $\{X_n\}$, let us reduce VAR(X) to its smallest values. Since the distribution of $\{G_n\}$ has been specified, its variance is fixed. Let $P(A_n = a) = 1$ and $P(B_n = b) = 1$. Then trivially $E(A) = a$, $E(B) = b$ and $VAR(X) = (a^2 + b^2)(\frac{1}{k})$. Under these conditions II.C.2.8 becomes

$$CORR(X_n, X_{n-1}) = \frac{ab}{a^2 + b^2}.$$

If we further specify that $a = b$, then $CORR(X_n, X_{n-1})$ achieves its maximum of 1/2. Q.E.D.

The situation developed above is comparable to the classical situation where the innovative factors have distribution, $N(0,\sigma^2)$. Except for the degenerate case where one coefficient is zero, the sequence generated by a linear combination of innovative factors may have the same first moment

as the innovative factors, but will have a different second moment. So, under comparable conditions the random, linear combination of Gamma(k,1) random variables can produce positive correlations equal in magnitude to the positive correlation produced by a classical normal process. The distribution of the $\{X_n\}$ under these circumstances is unknown.

If the distribution of the $\{X_n\}$ is constrained to be Gamma(k,1) and the Beta-Gamma generation scheme is used to generate the $\{X_n\}$, then the maximum correlation that can be achieved is one-fourth.

3. <u>Directional Moments</u>

As mentioned in II.B.5 the directional moments of a non-normal process are not necessarily equal and may provide valuable information about a time series. First, consider $E(X_n^2 X_{n-1})$. From II.C.1.1

$$X_n^2 = A_n^2 G_n^2 + 2A_n B_n G_n G_{n-1} + B_n^2 G_{n-1}^2$$

and

$$X_{n-1} = A_{n-1} G_{n-1} + B_{n-1} G_{n-2}$$

Therefore,

$$X_n^2 X_{n-1} = A_n^2 A_{n-1} G_n^2 G_{n-1} + 2A_n A_{n-1} B_n G_n G_{n-1}^2 + A_{n-1} B_n^2 G_{n-1}^3 + A_n B_{n-1} G_n^2 G_{n-2}$$

$$+ 2A_n B_n B_{n-1} G_n G_{n-1} G_{n-2} + B_n^2 B_{n-1} G_{n-1}^2 G_{n-2}$$

Taking expectations as in II.C.2 yields

$$E(X_n^2 X_{n-1}) = \frac{(k-q)^2(k-q+1)}{k^3} + \frac{2(k-q)^2(k+1)}{k^3} + \frac{(k-q)q(q+1)(k+2)}{k^4}$$

$$+ \frac{(k-q)(k-q+1)q}{k^3} + \frac{2(k-q)^2 q^2}{k^3} + \frac{q^2(q+1)}{k^3}$$

Upon simplification this produces

$$E(X_n^2 X_{n+1}) = \frac{(k-q)^2}{k^3}[3k-q+3] + \frac{(k-q)q}{k^4}\left[\frac{2q(k+1)+k+2}{k}\right]$$

$$+ \frac{q^2}{k^3}[2k-q+1] \qquad \text{(II.C.3.1)}$$

In an analogous fashion

$$X_n X_{n-1}^2 = A_n A_{n-1}^2 G_n G_{n-1}^2 + 2A_n A_{n-1} B_{n-1} G_n G_{n-1} G_{n-2} + A_n B_{n-1}^2 G_n G_{n-2}^2$$

$$+ A_{n-1}^2 B_n G_{n-1}^3 + 2A_{n-1} B_n B_{n-1} G_{n-1}^2 G_{n-2} + B_n B_{n-1}^2 G_{n-1} G_{n-2}^2$$

Taking expectations we have

$$E(X_n X_{n-1}^2) = \frac{(k-q)^2(k-q+1)}{k^3} + \frac{2(k-q)^2 q}{k^3} + \frac{(k-q)q(q+1)}{k^3}$$

$$+ \frac{(k-q)(k-q+2)q(k+2)}{k^4} + \frac{2(k-q)q^2(k+1)}{k^4} + \frac{q^2(q+1)}{k^3},$$

which simplifies to

$$E(X_n X_{n-1}^2) = \frac{(k-q)^2(k+q+1)}{k^3} + \frac{q(q+1)(2k-q)}{k^3}$$

$$+ \frac{(k-q)q(k+1)^2}{k^4} \qquad \text{(II.C.3.2)}$$

4. Empirical $P(X_{n+1} > X_n)$

No analytical procedure was found to determine $P(X_{n+1} > X_n)$. Hence, a simple computer program was constructed to evaluate this condition for a series of sixty-eight thousand pairs of numbers generated by the Beta-Gamma scheme for each of ten random number seeds. The answer obtained was considered to be accurate within 0.001. The comparisons were run for each of seventy-nine values of q, from 0.05 to 3.25 in increments of 0.05. All of the results of the comparisons fell in the range 0.499 to 0.501. Fourteen of the values were different from 0.500. No pattern was apparent in the deviations from 0.500 and these deviations were considered to be random fluctuations within the given margin for error. It thus seems clear that $P(X_{n+1} > X_n)$ for this process is like the GLAR(1) process but no proof has been found.

D. OTHER CASES OF THE GLARMA(p,q) MODEL

1. Introduction

A primary advantage of the GLARMA(p,q) model is the ease with which it can be adopted to cover a variety of special cases. Two special cases, the first-order autoregressive GLAR(1) and the first-order moving average GLMA(1), were

covered in II.B and II.C. The intention here is to briefly present three additional cases of the general model and derive the correlation structure of each case. The special cases considered are the first-order mixed model, GARMA(1,1), the second order autoregressive model GAR(2), and a bivariate, first-order, autoregressive model BGAR(1). The purpose in presenting these cases is to demonstrate the flexibility of GLARMA(p,q) and not to present a complete, detailed discussion of each model. Further extensions of the special cases of the GLARMA(p,q) model from the examples given are obvious. Details will not be given.

2. GLARMA(1,1)

Consider the following scheme for generating an $\{X_n\}$ sequence of random variables.

$$X_n = B_n A_{n-1} + C_n G_n, \tag{II.D.2.1}$$

$$A_n = D_n A_{n-1} + F_n G_n. \tag{II.D.2.2}$$

where $\{X_n, n = 1,2,\ldots\}$ is a second-order stationary sequence of Gamma(k,1) random variables; $\{A_n, n = 0,1,\ldots\}$ is a second-order stationary sequence of Gamma(k,1) random variables; $\{B_n, n = 1,2,\ldots\}$ is an iid sequence of Beta(k-q,q) random variables; $\{C_n, n = 1,2,\ldots\}$ is an iid sequence of Beta(q,k-q) random variables; $\{D_n, n = 1,2,\ldots\}$ is an iid sequence of Beta(k-r,r) random variables; $\{F_n, n = 1,2,\ldots\}$ is

an iid sequence of Beta(r,k-r) random variables; $\{B_n\}$, $\{C_n\}$, $\{D_n\}$, $\{F_n\}$, and $\{G_n\}$ are mutually independent; $0 < q \leq k$; $0 < r \leq k$. The Beta(m,n) density is given in II.B.1.3; the Gamma(k,1) density in II.B.1.2.

Before the serial correlation of the $\{X_n\}$ sequence can be determined, the serial correlation structure of the $\{A_n\}$ sequence and the correlation structure between the $\{A_n\}$ and $\{G_n\}$ sequences must be derived. Proceding first with the serial correlation of the $\{A_n\}$ sequence, from II.D.2.2 we have

$$A_n = D_n A_{n-1} + F_n G_n$$

So

$$A_n A_{n-1} = D_n A_{n-1}^2 + F_n G_n A_{n-1}$$

Using the iid nature of $\{D_n\}$, $\{F_n\}$, $\{G_n\}$; noticing that when $i > j$, D_i and A_j, F_i and A_j, and G_i and A_j are independent; recalling the independence of $\{F_n\}$, $\{G_n\}$; and taking expectations yields

$$E(A_n A_{n-1}) = \left(\frac{k-r}{k}\right)\frac{k(k+1)}{k^2} + \frac{r}{k} ,$$

$$E(A_n A_{n-1}) = \frac{k^2+k-r}{k^2} . \qquad \text{(II.D.2.3)}$$

Therefore,

$$COV(A_n, A_{n-1}) \quad = \quad \frac{k-r}{k^2}$$

and

$$CORR(A_n, A_{n-1}) \quad = \quad \frac{k-r}{k}. \qquad \text{(II.D.2.4)}$$

In fact A_n is just a GLAR(1) process, so the result is not surprising. Using II.D.2.2, II.D.2.3, and an induction argument leads to the general m-step correlation formula

$$CORR(A_n, A_{n-m}) \quad = \quad \left(\frac{k-r}{k}\right)^m, \quad n \geq m \geq 0. \qquad \text{(II.D.2.5)}$$

The correlation structure between $\{A_n\}$ and $\{G_n\}$ can be derived in a similar fashion. However, it is more direct to note that since the $\{A_n\}$ sequence is the same as the GLAR(1) process of Section II.B, the $\{A_n\}$, $\{G_n\}$ correlation structure will be the same as that derived for the $\{X_n\}$, $\{G_n\}$ sequences in II.B.2. Hence,

$$CORR(A_n, G_{n-m}) \quad = \quad \left(\frac{r}{k}\right)\left(\frac{k-r}{k}\right)^m, \quad n \geq m \geq 0. \qquad \text{(II.D.2.6)}$$

Of course, if $j > i$, then $CORR(A_i, G_j) = 0$

Now the serial correlation for the $\{X_n\}$ sequence can be found. From II.D.2.1

$$X_n = B_n A_{n-1} + C_n G_n$$

$$X_n X_{n-1} = (B_n A_{n-1} + C_n G_n)(B_{n-1} A_{n-2} + C_{n-1} G_{n-1})$$

$$= B_n B_{n-1} A_{n-1} A_{n-2} + B_n C_{n-1} A_{n-1} G_{n-1} + B_{n-1} C_n A_{n-2} G_n + C_n C_{n-1} G_n G_{n-1}.$$

Using the stationarity of the $\{A_n\}$ sequence, the iid nature and independence of $\{B_n\}$, $\{C_n\}$, $\{G_n\}$, the fact that A_i is independent of G_j when $j > i$, and the fact that $\{A_n\}$ is independent of $\{B_n\}$ by construction and taking expectations, we have

$$E(X_n X_{n-1}) = E(B_n)E(B_{n-1})E(A_{n-1}A_{n-2}) + E(B_n)E(C_{n-1})E(A_{n-1}C_{n-1}) + E(B_{n-1})E(C_n)E(A_{n-2})E(G_n) + E(C_n)E(C_{n-1})E(G_n)E(G_{n-1}),$$

$$= [E(B_n)]^2 E(A_{n-1}A_{n-2}) + E(B_n)E(C_{n-1})E(A_{n-1}G_{n-1}) + E(B_{n-1})E(C_n)E(A_{n-2})E(G_n) + [E(C_n)]^2 [E(G_n)]^2 . \quad \text{(II.D.2.7)}$$

From II.D.2.1

$$E(X_n)E(X_{n-1}) = [E(B_n)E(A_{n-1}) + E(C_n)E(G_n)] \times [E(B_{n-1})E(A_{n-2}) + E(C_{n-1})E(G_{n-1})],$$

$$E(X_n)E(X_{n-1}) = [E(B_n)]^2E(A_{n-1})E(A_{n-2})+2E(B_n)E(C_n)E(A_n)E(G_n) + [E(C_n)]^2[E(G_n)]^2. \qquad \text{(II.D.2.8)}$$

Using II.D.2.7 and II.D.2.8 to compute $COV(X_n,X_{n-1})$ yields

$$COV(X_n,X_{n-1}) = [E(B_n)]^2[E(A_{n-1}A_{n-2})-E(A_{n-1})E(A_{n-2})] + [E(B_n)E(C_n)][E(A_{n-1}G_{n-1})-E(A_n)E(G_n)]$$

Since $\{A_n\}$, $\{G_n\}$, and $\{X_n\}$ are all Gamma(k,1), $VAR(X_n) = VAR(A_n) = VAR(G_n)$. Hence,

$$CORR(X_n,X_{n-1}) = [E(B_n)]^2CORR(A_n,A_{n-1})+[E(B_n)E(C_n)]CORR(A_n,G_n)$$

From II.D.2.4 and II.D.2.6 we know this equals

$$CORR(X_n,X_{n-1}) = (\frac{k-q}{k})^2(\frac{k-r}{k}) + (\frac{k-q}{k})(\frac{q}{k})(\frac{r}{k}) \qquad \text{(II.D.2.9)}$$

Using II.D.2.5 and II.D.2.6 in an induction argument yields the general m-step correlation of

$$CORR(X_n,X_{n-m}) = (\frac{k-r}{k})^{m-1}[(\frac{k-q}{k})^2(\frac{k-r}{k})+(\frac{k-q}{k})(\frac{q}{k})(\frac{r}{k})].$$

Recognizing the expression in brackets as $CORR(X_n,X_{n-1})$ and letting ρ equal this correlation we have

$$\mathrm{CORR}(X_n, X_{n-m}) = \left(\frac{k-r}{k}\right)^{m-1}\rho, \quad n \geq m \geq 1. \qquad \text{(II.D.2.10)}$$

Figure II.D.2.1 shows the possible combinations of one- and two-step correlations for the GLARMA(1,1) model. This concludes the development of the correlation structure of the GLARMA(1,1) model.

3. GLAR(2)

Jacobs and Lewis [Ref. 12] first developed the following mixture scheme for generating a p-order autoregressive processes. We now adopt that scheme for generating a second-order autoregressive sequence of random variables. This is the special case of GLARMA(p,q) with p = 2 and q = 0. As such it closely resembles the GLAR(1) process. Let

$$X_n = B_n X_{n-T_n} + C_n G_n, \qquad \text{(II.D.3.1)}$$

where $\{X_n, n = -1,0,1,\ldots\}$ is assumed to be a second-order stationary sequence of Gamma(k,1) random variables; $\{B_n, n = 1,2,\ldots\}$ is an iid sequence of Beta(k-q,q) random variables; $\{C_n\}$ is an iid sequence of Beta(q,k-q) random variables; $\{G_n\}$ is an iid sequence of Gamma(k,1) random variables; $\{B_n\}$, $\{C_n\}$, $\{G_n\}$ are independent; also T_n is iid with $P(T_n = 1) = 1 - P(T_n = 2) = \alpha$. The Gamma(k,1) and Beta(m,n) densities are found in II.B.1.2 and II.B.1.3, respectively.

This generation scheme works even though X_{n-1} and X_{n-2} are dependent random variables. The mixture of X_{n-1} and

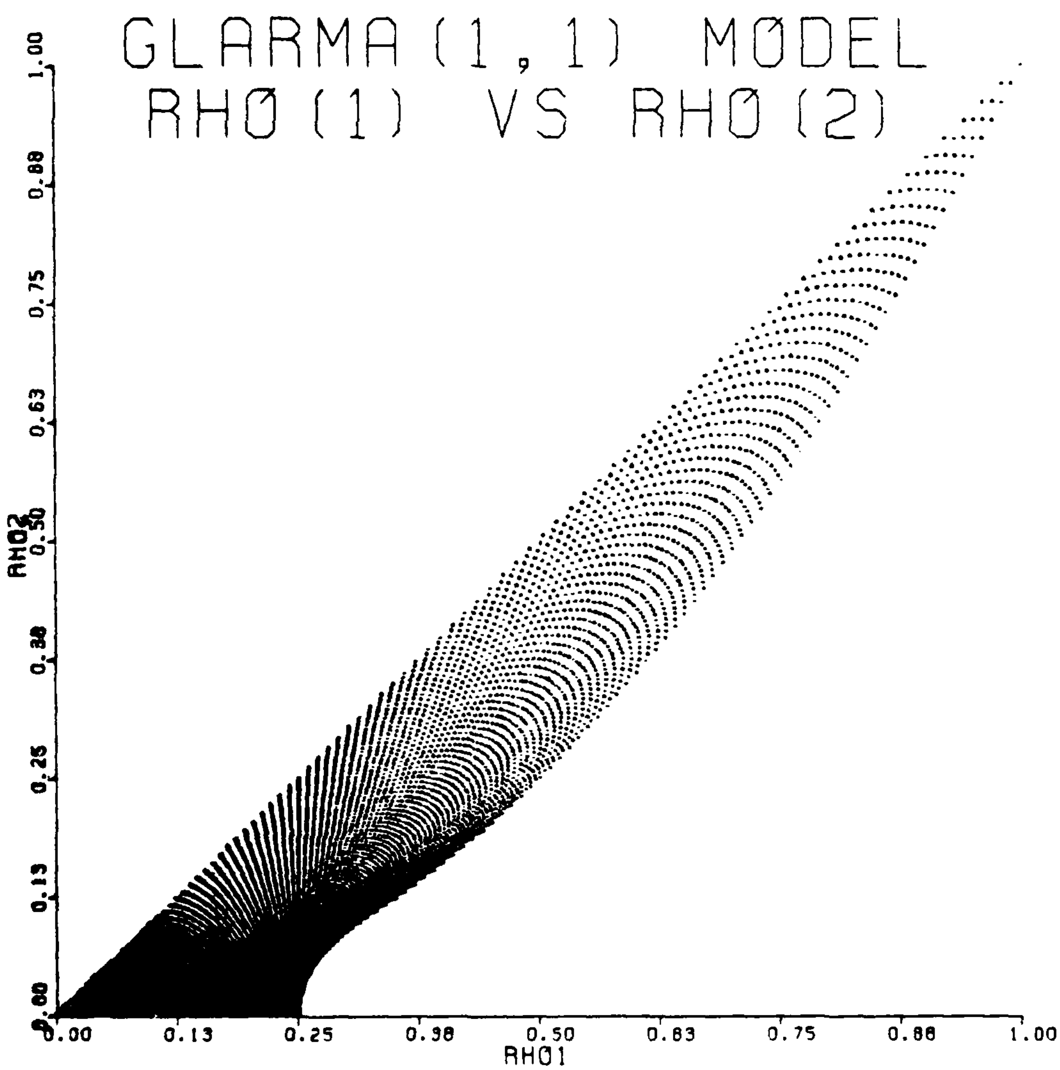


Figure II.D.2.1

X_{n-2} produced by II.D.3.1 is Gamma distributed and independent of G_n. Hence, II.D.3.1 is simply another example of the random sum of two independent Gamma random variables producing another Gamma random variable.

Two special cases of II.D.3.1 are as follows. When $\alpha = 1$, the GLAR(2) process reduces to the GLAR(1). When $q = k$, the $\{X_n\}$ sequence is iid.

The serial correlation of the $\{X_n\}$ sequence can be calculated in the usual fashion, assuming stationarity of the process we have

$$X_n = B_n X_{n-T_n} + C_n G_n \qquad k > 0; \quad 0 < q \leq k$$

and

$$X_n X_{n-1} = B_n X_{n-T_n} X_{n-1} + C_n G_n X_{n-1}.$$

Using the independence of $\{C_n\}$ and $\{G_n\}$, the fact that X_i is independent of C_j, G_j and B_j when $j > i$ and taking expectations we have

$$\begin{aligned} E(X_n X_{n-1}) &= \alpha E(B_n) E(X_{n-1}^2) + (1-\alpha) E(B_n) E(X_{n-1} X_{n-2}) + E(C_n) E(G_n) E(X_{n-1}) \\ &= \alpha\left(\frac{k-q}{k}\right)\left(\frac{k[k+1]}{k^2}\right) + (1-\alpha)\left(\frac{k-q}{k}\right) E(X_{n-1} X_{n-2}) + \frac{q}{k}, \end{aligned}$$

since $E(X_n) = E(G_n) = 1$ by assumption. Using the second-order

stationarity of $\{X_n\}$

$$E(X_n X_{n-1})\left(1 - \frac{(1-\alpha)(k-q)}{k}\right) = \frac{\alpha(k-q)(k+1)}{k^2} + \frac{q}{k}$$

Upon simplification

$$E(X_n X_{n-1}) = \frac{\alpha k^2+\alpha k-\alpha kq-\alpha q+kq}{k(q+\alpha k-\alpha q)}$$

Hence,

$$COV(X_n, X_{n-1}) = \left(\frac{1}{k}\right)\left[\frac{\alpha(k-q)}{q+\alpha(k-q)}\right]$$

and

$$CORR(X_n, X_{n-1}) = \frac{\alpha(k-q)}{q+\alpha(k-q)} \qquad \text{(II.D.3.2)}$$

If $\alpha = 1$, this equals $1 - \frac{q}{k}$; if $k = q$ then it is zero. Since $q \geq k$ it is clearly non-negative.

The lag two correlation can be calculated in a similar fashion. We have

$$X_n = B_n X_{n-T_n} + C_n G_n;$$

$$X_n X_{n-2} = B_n X_{n-T} X_{n-2} + C_n G_n X_{n-2}$$

and

$$E(X_n X_{n-2}) = \alpha(\frac{k-q}{k})E(X_{n-1}X_{n-2}) + (1-\alpha)(\frac{k-q}{k})E(X_{n-2}^2) + \frac{q}{k}.$$

Using the second-order stationarity of the $\{X_n\}$ sequence we can write

$$E(X_n)E(X_{n-2}) = \alpha(\frac{k-q}{q})E(X_{n-1})E(X_{n-2}) + (1-\alpha)(\frac{k-q}{k})[E(X_{n-2})]^2$$

$$+ \frac{q}{k}[E(X_n)]^2,$$

so that

$$\begin{aligned}
COV(X_n, X_{n-2}) &= \alpha(\frac{k-q}{k})[E(X_{n-1}X_{n-2}) - E(X_{n-1})E(X_{n-2})] \\
&\quad + (1-\alpha)(\frac{k-q}{k})[E(X_{n-2}^2) - \{E(X_{n-2})\}^2] \\
&\quad + \frac{q}{k} - \frac{q}{k}[E(X_n)]^2 \\
&= \alpha(\frac{k-q}{k})[E(X_{n-1}X_{n-2}) - E(X_{n-1})E(X_{n-2})] \\
&\quad + \frac{(1-\alpha)(k-q)}{k}[E(X_{n-2}^2) - \{E(X_{n-2})\}^2] \\
&= \alpha(\frac{k-q}{k})COV(X_{n-1}, X_{n-2}) + \frac{(1-\alpha)(k-q)}{k}VAR(X_{n-2}).
\end{aligned}$$

Hence,

$$CORR(X_n, X_{n-2}) = \alpha(\frac{k-q}{k})CORR(X_{n-1}, X_{n-2}) + \frac{(1-\alpha)(k-q)}{k},$$

$$\mathrm{CORR}(X_n, X_{n-2}) = \left(\frac{k-q}{k}\right)\left[\frac{q+\alpha k-2\alpha q}{q+\alpha(k-q)}\right] \qquad \text{(II.D.3.3)}$$

The solution procedure for II.D.3.3, if followed for X_{n-m}, will produce the general recursion equation (Yule-Walker) that can be used along with II.D.3.2 and II.D.3.3 to compute the m-step correlation. The formula thus produced is

$$\begin{aligned}\mathrm{CORR}(X_n, X_{n-m}) &= \left(\frac{k-q}{k}\right)[\alpha\,\mathrm{CORR}(X_n, X_{n+1-m}) \\ &\quad + (1-\alpha)\,\mathrm{CORR}(X_n, X_{n+2-m})], \qquad \text{(II.E.3.4)} \\ &\quad n \geq m \geq 1.\end{aligned}$$

As mentioned in previous sections the (X_n, G_n) pair can be considered to be a correlated, bivariate pair of Gamma(k,1) random variables. Therefore, we proceed to derive the correlation structure between these two sequences. From II.D.3.1 we have

$$X_n = B_n X_{n-T_n} + C_n G_n$$

and

$$X_n G_n = B_n X_{n-T_n} G_n + C_n G_n^2.$$

Recalling the independence of $\{C_n\}$ and $\{G_n\}$ and X_i and G_j when $j > i$, taking expectations yields

$$E(X_n G_n) = \left(\frac{k-q}{k}\right) + \left(\frac{q}{k}\right)\frac{(k)(k+1)}{k^2}$$

$$= \frac{k^2+q}{k^2}$$

Hence,

$$COV(X_n, G_n) = \frac{q}{k^2}$$

and

$$CORR(X_n, G_n) = \frac{q}{k}. \qquad \text{(II.D.3.5)}$$

Continuing this process we have

$$X_n = B_n X_{n-T} + C_n G_n$$

and

$$X_n G_{n-1} = B_n X_{n-T} G_{n-1} + C_n G_n G_{n-1}.$$

Thus

$$E(X_n G_{n-1}) = \alpha\left(\frac{k-q}{k}\right) E(X_{n-1} G_{n-1}) + (1-\alpha)\left(\frac{k-q}{k}\right) + \frac{q}{k}.$$

Hence,

$$COV(X_n, G_{n-1}) = \alpha\left(\frac{k-q}{k}\right) COV(X_{n-1}, G_{n-1})$$

And

$$CORR(X_n, G_{n-1}) = \alpha\left(\frac{k-q}{k}\right)\left(\frac{q}{k}\right).$$

One further step in this process using an arbitrary m-step lag produces the general recursion formula

$$CORR(X_n, G_{n-m}) = \left(\frac{k-q}{k}\right)[\alpha CORR(X_n, G_{n+1-m}) + (1-\alpha) CORR(X_n, G_{n+2-m})], \quad n \geq m \geq 2 \quad \text{(II.D.3.7)}$$

Figure II.D.3.1 displays a plot of the possible combinations of $CORR(X_n, X_{n-1})$ and $CORR(X_n, X_{n-2})$. Note that when $\alpha = 1$, the GLAR(2) process reduces to the GLAR(1) and $CORR(X_n, X_{n-2}) = \left(\frac{k-q}{k}\right)^2$ which defines the lower boundary of the plot. When $\alpha = 0$, $CORR(X_n, X_{n-1}) = 0$ and $CORR(X_n, X_{n-2}) = \frac{k-q}{k}$ which goes from zero to one. In the interior of the graph, when α does not assume an extreme value, $CORR(X_n, X_{n-2})$ does not reach a value of one. This is demonstrated by the following calculation. From II.D.3.3

$$CORR(X_n, X_{n-2}) = \left(\frac{k-q}{k}\right)\left[\frac{q+\alpha k-2\alpha q}{q+\alpha(k-q)}\right].$$

If this correlation is to equal one, then

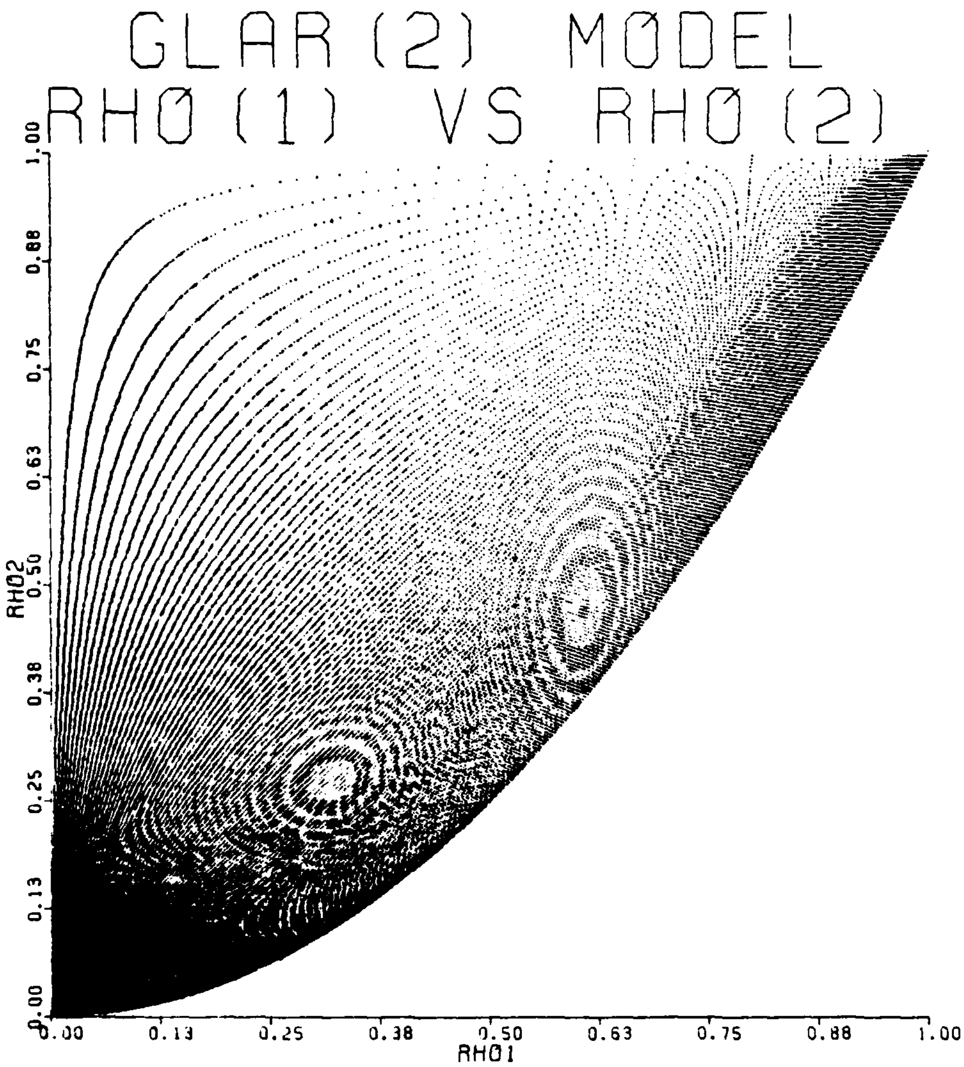


Figure II.D.3.1

$$\frac{q+\alpha k-2\alpha q}{q+\alpha(k-q)} = \frac{k}{k-q} \tag{II.D.3.8}$$

which quickly reduces to

$$q(-2\alpha k-q+2kq) = 0$$

Since $0 < q$, this requires

$$-2\alpha k-q+2\alpha q = 0$$

Therefore,

$$\alpha = \frac{q}{2(q-k)}.$$

Since we know that $0 < q \leq k$, α must be less than zero. However, α is a probability and, hence, is non-negative. Therefore, the original requirement in II.D.3.8 cannot be satisfied. Hence, $CORR(X_n, X_{n-2})$ cannot equal one.

4. <u>BGAR(1), Bivariate Model</u>

To this point the only examples of bivariate Gamma processes presented were those in which the innovation sequence was one part of the bivariate process with the generated sequence the other part. The simple random, linear structure of the GLAR(1) process makes it easily extendable to a variety of bivariate models. We address only the simplest. Consider the following pair of random variables, both of which are formed from the same innovation process, $\{G_n\}$:

$$X_n = B_n X_{n-1} + C_n G_n, \quad \text{(II.D.4.1)}$$

$$Y_n = D_n Y_{n-1} + F_n G_n, \quad \text{(II.D.4.2)}$$

where $\{X_n, n = 0,1,\ldots\}$ is a second-order stationary sequence of Gamma$(k,1)$ random variables; $\{Y_n, n = 0,1,\ldots\}$ is a second-order stationary sequence of Gamma$(k,1)$ random variables; $\{B_n, n = 1,2,\ldots\}$ is an iid sequence of Beta$(k-q,q)$ random variables; $\{C_n, n = 1,2,\ldots\}$ is an iid sequence of Beta$(q,k-q)$ random variables; $\{D_n, n = 1,2,\ldots\}$ is an iid sequence of Beta$(k-r,r)$ random variables; $\{F_n, n = 1,2,\ldots\}$ is an iid sequence of Beta$(r,k-r)$ random variable; $\{G_n, n = 1,2,\ldots\}$ is an iid sequence of Gamma$(k,1)$ random variables; $\{B_n\}$, $\{C_n\}$, $\{D_n\}$, $\{F_n\}$, and $\{G_n\}$ are independent; $0 < r \leq k$; $0 < q \leq k$.

This is a special case of a general situation. In a more general case the $\{X_n\}$ and $\{Y_n\}$ sequences could have separate, correlated innovation sequences instead of sharing a single $\{G_n\}$ sequence. In addition, the $\{B_n\}$ and $\{D_n\}$ sequences and the $\{C_n\}$ and $\{F_n\}$ sequences could be correlated.

Before examining the correlation between $\{X_n\}$ and $\{Y_n\}$, it will be necessary to address the correlation of each of these sequences with the $\{G_n\}$ sequence. This is most easily handled by recognizing that the relationship of the $\{X_n\}$ and $\{Y_n\}$ sequences to the $\{G_n\}$ sequence is exactly the same as the $\{A_n\}$ sequence to the $\{G_n\}$ sequence in II.D.2.2.

Hence, the correlation structure will be the same. Therefore, if we let $\rho_X = \text{CORR}(X_n, X_{n-1})$ and $\rho_Y = \text{CORR}(Y_n, Y_{n-1})$, $n = 1,2,\ldots$, these correlation structures can be written without further analysis as

$$\text{CORR}(X_n, G_n) = \frac{q}{k} = 1 - \text{CORR}(X_n, X_{n-1}) = 1 - \rho_X; \qquad \text{(II.D.4.3)}$$

$$\text{CORR}(X_n, G_{n-1}) = \left(\frac{q}{k}\right)\left(\frac{k-q}{k}\right) = (1 - \rho_X)\rho_X; \qquad \text{(II.D.4.4)}$$

$$\text{CORR}(X_n, G_{n-m}) = \left(\frac{q}{k}\right)\left(\frac{k-q}{k}\right)^m = (1 - \rho_X)\rho_X^m, \; n \geq m \geq 0. \qquad \text{(II.D.4.5)}$$

and

$$\text{CORR}(Y_n, G_n) = \frac{r}{k} = 1 - \text{CORR}(Y_n, Y_{n-1}) = 1 - \rho_Y; \qquad \text{(II.D.4.6)}$$

$$\text{CORR}(Y_n, G_{n-1}) = \left(\frac{r}{k}\right)\left(\frac{k-r}{k}\right) = (1 - \rho_Y)\rho_Y, \qquad \text{(II.D.4.7)}$$

$$\text{CORR}(Y_n, G_{n-m}) = \left(\frac{r}{k}\right)\left(\frac{k-r}{k}\right)^m = (1 - \rho_Y)\rho_Y^m, \; n \geq m \geq 0 \qquad \text{(II.D.4.8)}$$

The assumption is that the bivariate process is stationary, although it should be noted that starting the univariate processes in a stationary mode does not make the bivariate process stationary. The initial pair $\{X_0, Y_0\}$ must have the bivariate Gamma distribution associated with the stationary Markovian process.

Now we address the cross correlation between $\{X_n\}$ and $\{Y_n\}$. Start with II.D.4.1. We have

$$X_n = B_n X_{n-1} + C_n G_n$$

and

$$X_n Y_{n-1} = B_n X_{n-1} Y_{n-1} + C_n G_n Y_{n-1}.$$

Using the independence of $\{C_n\}$ and $\{G_n\}$ and that of Y_i and G_j and C_j when $j > i$ and taking expectations

$$E(X_n Y_{n-1}) = E(B_n) E(X_{n-1} Y_{n-1}) + E(C_n) E(G_n) E(Y_{n-1}),$$

so that

$$E(X_n Y_{n-1}) = \left(\frac{k-q}{k}\right) E(X_{n-1} Y_{n-1}) + \frac{q}{k}. \qquad \text{(II.D.4.9)}$$

Now starting with II.E.4.2, we have

$$Y_n = D_n Y_{n-1} + F_n G_n$$

and

$$X_n Y_n = D_n Y_{n-1} X_n + F_n G_n X_n.$$

Taking expectations as above

$$E(X_n Y_n) = E(D_n) E(X_n Y_{n-1}) + E(F_n) E(G_n X_n).$$

Using II.D.4.3 we deduce $E(G_n X_n) = \frac{k^2+q}{k^2}$ so that

$$E(X_n Y_n) = (\frac{k-r}{k}) E(X_n Y_{n-1}) + (\frac{r}{k})(\frac{k^2+q}{k^2}). \qquad \text{(II.D.4.10)}$$

Invoking the second-order stationarity of the $\{X_n, Y_n\}$ sequences, using II.D.4.9 and II.D.4.10 and letting $w = E(X_i Y_i)$ and $z = E(X_i Y_{i-1})$, we have the two equations

$$w = (\frac{k-r}{k}) z + (\frac{r}{k})(\frac{k^2+q}{k^2}), \qquad \text{(II.D.4.11)}$$

$$z = (\frac{k-q}{k}) w + \frac{q}{k}. \qquad \text{(II.D.4.12)}$$

Using II.D.4.12 to substitute for z in II.E.4.11 yields

$$w = (\frac{k-r}{k}) [(\frac{k-q}{k}) w + \frac{q}{k}] + (\frac{r}{k})(\frac{k^2+q}{k^2})$$

$$= [\frac{k^2-kr-kq+qr}{k^2}] w + \frac{(k-r)q}{k^2} + \frac{(k^2+q)r}{k^3}$$

When $E(X_n Y_n)$ is substituted for w after simplification, this produces

$$E(X_n Y_n) = \frac{k^2 q - kqr + k^2 r + rq}{k(kq+kr-qr)}. \qquad \text{(II.D.4.13)}$$

Hence

$$COV(X_n, Y_n) = \frac{rq}{k(kq+kr-qr)}$$

and

$$\mathrm{CORR}(X_n,Y_n) = \frac{rq}{kq+kr-qr}, \quad k > 0,\ 0 < q \le k,\ 0 < r \le k. \tag{II.D.4.14}$$

$$= \frac{(1-\rho_X)(1-\rho_Y)}{(1-\rho_X)+(1-\rho_Y)\rho_X}$$

$$= \frac{(1-\rho_X)(1-\rho_Y)}{(1-\rho_Y)+(1-\rho_X)\rho_Y}. \tag{II.D.4.15}$$

This latter expression follows since for a given k the correlation structure is parameterized by r and q or equivalently by the serial correlations ρ_X and ρ_Y given at II.D.4.3 and II.D.4.6.

We can now substitute II.E.4.13 into D.E.4.12 to solve for $E(X_nY_{n-1})$.

$$E(X_nY_{n-1}) = \left(\frac{k-q}{k}\right)\left[\frac{k^2q-kqr+k^2r+rq}{k(kq+kr-qr)}\right] + \frac{q}{k}$$

$$= \frac{k^3q+k^3r+kqr-k^2qr-q^2r}{k^2(kq+kr-qr)}.$$

Hence,

$$\mathrm{COV}(X_n,Y_{n-1}) = \frac{kqr-q^2r}{k^2(kq+kr-qr)}$$

and

$$\mathrm{CORR}(X_n,Y_{n-1}) = \left(\frac{qr}{kq+kr-qr}\right)\left(\frac{k-q}{k}\right) = \mathrm{CORR}(X_nY_n)\rho_X \tag{II.D.4.16}$$

Continuing in this vein produces the general formula

$$\mathrm{CORR}(X_n, Y_{n-m}) = \left(\frac{qr}{kq+kr-qr}\right)\left(\frac{k-q}{k}\right)^m \tag{II.D.4.17}$$

$$= \mathrm{CORR}(X_n, Y_n)\,\rho_X^m, \quad n \geq m \geq 0.$$

Solving for correlations where the $\{X_n\}$ lags the $\{Y_n\}$ sequence is similar to the above process, but somewhat abbreviated. Starting with II.D.4.2,

$$Y_n = D_n Y_{n-1} + F_n G_n,$$

we have

$$Y_n X_{n-1} = D_n Y_{n-1} X_{n-1} + F_n G_n X_{n-1}$$

Taking expectations as before gives

$$E(Y_n X_{n-1}) = E(D_n) D(Y_{n-1} X_{n-1}) + E(F_n) E(G_n) E(X_{n-1})$$

Using the second-order stationarity of $\{X_n\}$ and $\{Y_n\}$, $E(Y_{n-1}X_{n-1})$ is known from II.E.4.13, so

$$E(Y_n X_{n-1}) = \left(\frac{k-r}{k}\right)\left(\frac{k^2q-kqr+k^2r+rq}{k[kq+kr-qr]}\right) + \frac{r}{k}$$

$$= \frac{k^3q+k^3r+kqr-k^2qr-qr^2}{k^2(kq+kr-qr)}.$$

Hence,

$$COV(Y_n, Y_{n-1}) = \frac{kqr - qr^2}{k^2(kq+kr-qr)}$$

and

$$CORR(Y_n, X_{n-1}) = \left(\frac{qr}{kq+kr-qr}\right)\left(\frac{k-r}{k}\right) = CORR(X_n Y_n)\rho_Y.$$

Further computations of this nature produce the general formula

$$CORR(Y_n, X_{n-m}) = \left(\frac{qr}{kq+kr-qr}\right)\left(\frac{k-r}{k}\right)^m$$

$$= CORR(X_n, Y_n)\rho_Y^m \qquad n \geq m \geq 0;\ k > 0;\ 0 < q \leq k;\ 0 < r \leq k.$$

To examine the correlation further, note that if $\rho_X = \rho_Y = \rho$, then II.D.4.15 yields

$$CORR(X_n, Y_n) = \frac{1-\rho}{1+\rho}.$$

Thus if $\rho = 0$ (i.e., the $\{X_n\}$ and $\{Y_n\}$ processes are iid sequences), this correlation is one. For $\{X_n\}$ and $\{Y_n\}$ to be iid, we must have $X_n = G_n = Y_n$. Therefore, this limiting correlation does make sense and suggests that perhaps a bivariate Gamma should be used as the innovation process. This

would allow for separate control of the serial correlation (auto-correlation) and cross-correlation of the $\{X_n\}$ and $\{Y_n\}$ sequences.

If $\rho \to 1$ (i.e., $X_n \simeq X_{n-1}$, $Y_n \simeq Y_{n-1}$), the effect of the innovation sequence is slight and the cross-correlation between $\{X_n\}$ and $\{Y_n\}$ goes to zero. In a more complicated model than we have addressed here the cross-correlation may be controlled by imposing a correlation on the $\{B_n\}$ and $\{D_n\}$ sequences.

Cross-coupled processes, as discussed in Gaver and Lewis [Ref. 2], are possible. These can be used to create negative serial correlations in the $\{X_n\}$ and $\{Y_n\}$ processes.

E. NUMERICAL CONVERGENCE OF THE MAXIMUM LIKELIHOOD COMPUTER PROGRAM AND SIMULATION STUDY OF PROPERTIES OF ESTIMATORS

The program described in Section II.B.4 for computing the conditional density function in the GLAR1 process was used in two fashions. First, it was tested by using computer generated data from a GLAR(1) process with known parameter values k, q, and μ. Simultaneously a simulation of properties of m.l.e*'s $\hat{k}$ and $\hat{q}$ for k and q was conducted. Second, it was used to estimate the parameters in the GLAR(1) model for real wind speed data. Only the first use is covered in this section. The second use is addressed in Chapter IV, Preliminary Data Analysis.

Four aspects of the program were addressed in the use of the program with simulated data. These were: sensitivity of the maximum seeking method to start point, the size of the

*maximum likelihood estimate

standard deviation of the m.l.e and moment estimates of k and q produced, and the degree of bias, if any, in the estimates. In addition, normality of the distributions of the estimates was investigated using normal plots and Shapiro-Wilks tests.

Because of the large computation time involved in obtaining a m.l.e.* the simulation study was small. Three types of data generated from GLAR(1) processes were used to exercise the program. Each type of data consisted of ten independent sets (replications) of 1000 data points each. The k and q values and the correlation were varied from one type of data to another. Thus the first type of data was generated with a k value of 4.0 and a q value of 1.0. These parameter values produce a correlation of 0.75 (see equation II.B.2.1). The second type of data varied the correlation, but retained the same k value. A k of 4.0 and a q of 3.0 produce a correlation of 0.25. These values were used to produce data sets of the second type. The third type of data returned to a high correlation, but used a small k value. The parameter values used to generate this data type were a k of 0.75 and a q of 0.1875. These values also produce a correlation of 0.75. Table II.E.1 summarizes these cases. In all cases, $\mu = 1$.

TABLE II.E.1

CASE	k	q	ρ
1	4.0	1.0	0.75
2	4.0	3.0	0.25
3	0.75	0.1875	0.75

*maximum likelihood estimates

The Gamma variates were generated by the program LLRANDOMII [Ref. 16] and all runs were performed on the NPS IBM/3033 computer.

To test the sensitivity of the maximum likelihood computer program to the starting point of the search for a maximum, each set of data was used in two runs of the program. The first run used the actual parameter values k and q as a start point. The second run used the moment estimates $\tilde{k}$ and $\tilde{q}$ of the parameter values (see equations II.B.4.15 and II.B.4.16) as a start point. The resulting m.l.e.* parameter estimates $\hat{k}$ and $\hat{q}$ were recorded.

The first case had k = 4.0 and q = 1.0. The results of the computer runs are presented in Table II.E.2 for data of the first type. The last column presents the two-dimensional distance in the (k,q) plane between the estimates produced by the two different start points for each data set. Although the values do not differ widely, the relatively large differences in some cases indicate that the likelihood function is relatively flat near the maximum.

However, there is another factor which may be contributing to differences in final parameter estimates. The calculation of the likelihood function for 1000 data points requires the numerical evaluation of 999 integrals (see equations II.B.4.2 and II.B.4.3). The IMSL routine DCADRE was used for this evaluation. Two of the parameters in the call to DCADRE are

---

*maximum likelihood estimates.

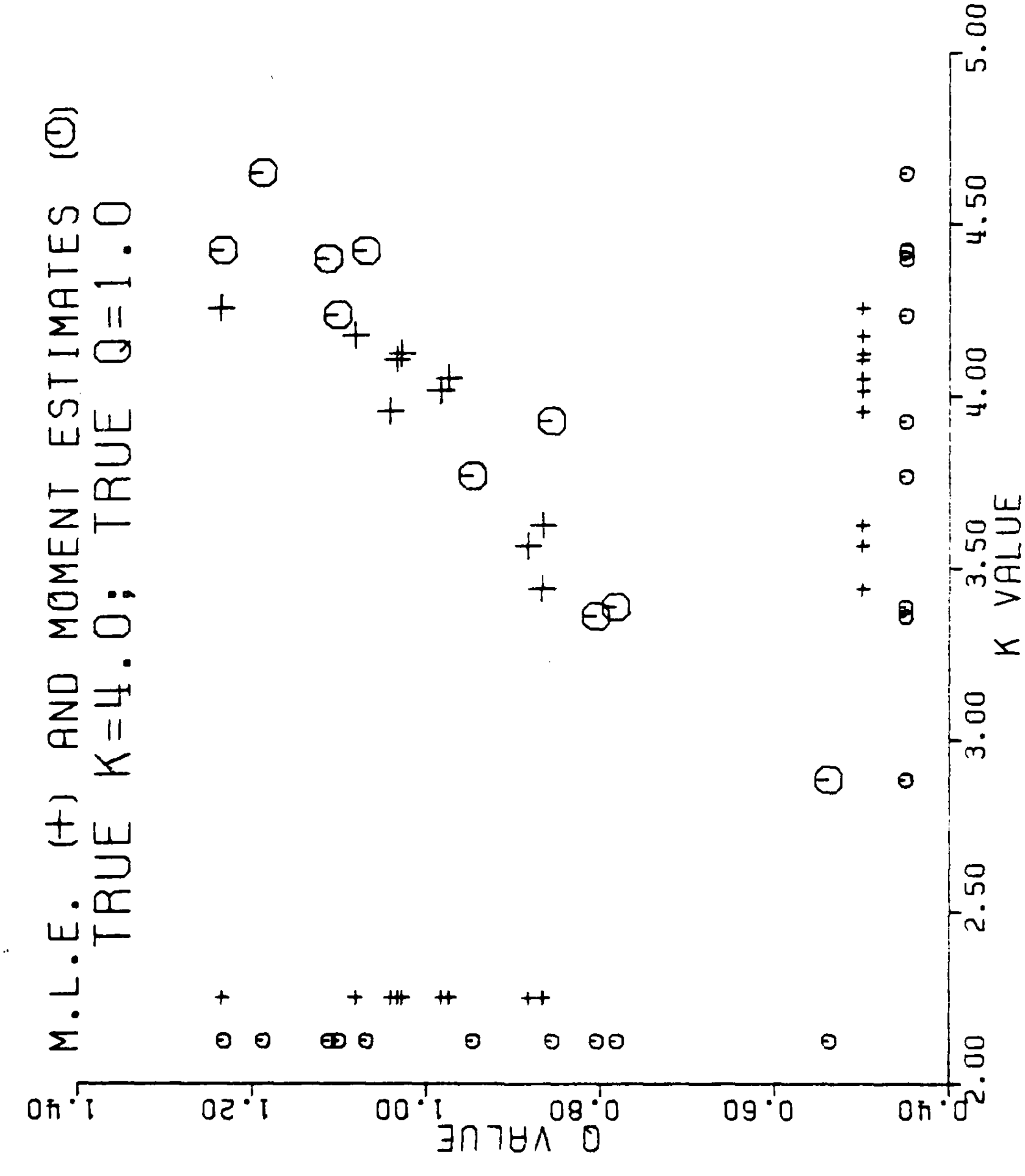
TABLE II.E.2

Results of Search for Maximum Likelihood Estimates in a GLAR(1) Model; $k = 4.0$, $q = 1.0$

Data Set	Starting Value k	Starting Value q	Run Time (in Min.)	Number of Iterations	Ending Value k	Ending Value q	Maximum Likelihood	Value of Difference (10^{-3})
0	4.000	1.000	20	6	3.668	0.882	-214.479	46
0	2.884	0.538	129	17	3.624	0.886	-214.488	
1	4.000	1.000	19	2	4.004	0.979	-204.819	10
1	4.235	1.101	46	5	4.014	0.983	-204.820	
2	4.000	1.000	104	14	3.451	0.869	-260.332	11
2	3.360	0.805	112	14	3.440	0.867	-260.333	
3	4.000	1.000	16	2	3.977	1.051	-206.416	24
3	3.767	0.947	71	15	3.955	1.041	-206.418	
4	4.000	1.000	52	5	4.246	1.232	-304.365	8
4	4.423	1.233	45	6	4.254	1.235	-304.366	
5	4.000	1.000	47	3	4.061	0.998	-211.074	68
5	4.647	1.187	27	4	4.122	1.028	-211.060	
6	4.000	1.000	61	4	3.593	0.891	-229.087	29
6	3.387	0.782	55	9	3.565	0.883	-229.092	
7	4.000	1.000	37	5	4.041	0.973	-241.637	10
7	4.421	1.069	82	7	4.051	0.974	-241.636	
8	4.000	1.000	37	6	4.024	0.999	-240.432	88
8	3.927	0.856	24	8	4.106	1.033	-240.424	
9	4.000	1.000	51	3	4.109	1.054	-255.245	70
9	4.398	1.112	38	5	4.174	1.081	-255.229	

the relative and absolute errors allowed for this calculation. Practical considerations of computer run time dictated relatively large values of 10^{-4} for each of these parameters. This error when applied 999 times in the calculation of the likelihood function may have lead to the differences in m.l.e. parameter estimates. As a test of this hypothesis the data set which produced the largest distance between the pairs of estimates (data set 8) was rerun with DCADRE error parameters set at 10^{-10}. The run which used the actual parameter values as a start point ran in 171 minutes and produced estimates of: $\hat{k} = 4.102$, $\hat{q} = 1.031$. The run which used the moment estimates as a start point was terminated after 404 minutes CPU time. At that point it had estimates of: $\hat{k} = 4.088$, $\hat{q} = 1.025$. The distance of 15×10^{-3} represents a significant reduction in the previous distance of 88×10^{-3}. It seems from this example that the program can be made less sensitive to the starting point by increasing the accuracy with which DCADRE computes the integrals in the likelihood function. Of course, a considerable increase in computational cost is incurred. This cost is not practical in these simulations or necessary since only a rough idea of the properties of the estimates was sought.

The results of the runs for data type one are also presented in Figure II.E.1. First, each pair of estimates, $(\tilde{k},\tilde{q})$ and $(\hat{k},\hat{q})$ is plotted in the k,q plane. Then each point is projected along each axis to more conveniently reflect the marginal variation in the estimates for each parameter.



M.L.E. (+) AND MOMENT ESTIMATES (Θ)
TRUE K=4.0; TRUE Q=1.0
Q VALUE
K VALUE
1.40
1.20
1.00
0.80
0.60
0.40
2.00
2.50
3.00
3.50
4.00
4.50
5.00

Figure II.E.1

When $k = 4.0$ and $q = 1.0$, the method of moments produces estimates for k with a sample mean of 3.94 and a sample standard deviation of 0.55. The statistics for the estimates of q are a sample mean of 0.96 and a sample standard deviation of 0.21. The maximum likelihood method produces estimates of k with a sample mean of 3.93 and a sample standard deviation of 0.27. The values for q estimates are a sample mean of 1.00 and a sample standard deviation of 0.11. Although the method of moments and maximum likelihood method do not produce significantly different values for the mean of the estimates of the parameters, the lower estimated standard deviation for the likelihood method makes this technique more desirable from the standpoint of precision. No bias was evident in either estimation technique with the precision attained in the simulations.

The second case was the low correlation case with $k = 4.0$ and $q = 3.0$. Here the distinction between the two estimation procedures is not as clear (Table II.E.3 and Figure II.E.2). The method of moments produced estimates for k with a sample mean of 3.99 and sample standard deviation of 0.17. The estimates for q had a sample mean of 2.9[illegible] and sample standard deviation of 0.18. The maximum likelihood method produced estimates of k which had sample mean 4.00 and sample standard deviation 0.16. The q estimates had a sample mean of 2.99 and sample standard deviation of 0.17. It is clear that neither method of parameter estimation enjoys a distinct advantage

TABLE II.E.3

Results of Search for Maximum Likelihood Estimates in a GLAR(1) Model; $k = 4.0$, $q = 3.0$

Data Set	Starting Value k	Starting Value q	Run Time (in Min.)	Number of Iterations	Ending Value k	Ending Value q	Maximum Likelihood	Value of Difference (10^{-3})
0	4.000	3.000	38	6	4.078	3.187	-620.637	8
0	4.019	3.271	25	5	4.085	3.192	-620.637	
1	4.000	3.000	23	4	3.780	2.934	-655.488	5
1	3.718	2.843	26	4	3.775	2.931	-655.489	
2	4.000	3.000	40	3	3.789	2.765	-663.886	3
2	3.912	2.862	27	5	3.789	2.762	-663.886	
3	4.000	3.000	44	7	4.321	3.132	-585.776	5
3	4.000	3.116	50	4	4.316	3.130	-585.774	
4	4.000	3.000	44	4	4.166	3.403	-604.719	22
4	3.958	3.174	68	6	4.152	3.385	-600.715	
5	4.000	3.000	27	6	3.890	2.904	-638.667	5
5	4.001	3.048	13	4	3.885	2.902	-638.666	
6	4.000	3.000	43	3	4.051	2.873	-599.581	5
6	4.221	2.902	49	5	4.049	2.878	-599.581	
7	4.000	3.000	23	2	3.954	2.934	-627.062	11
7	4.079	3.074	16	7	3.963	2.941	-627.058	
8	4.000	3.000	21	4	3.917	2.898	-637.234	11
8	3.708	2.633	32	6	3.908	2.891	-637.233	
9	4.000	3.000	15	2	4.111	2.924	-590.563	14
9	4.093	2.906	43	2	4.121	2.934	-590.562	

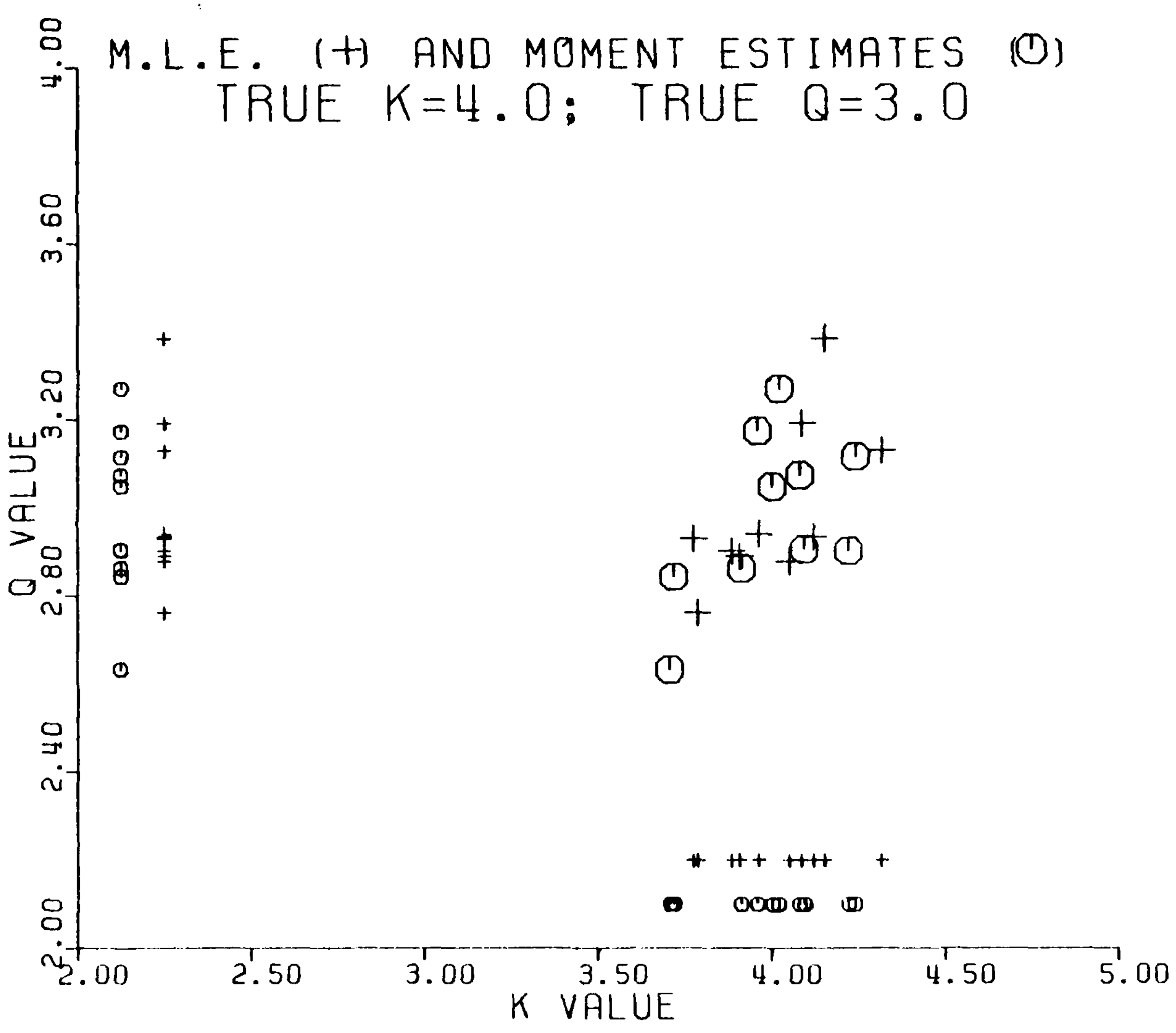

M.L.E. (+) AND MØMENT ESTIMATES (⊖)
TRUE K=4.0; TRUE Q=3.0
Q VALUE
4.00
3.60
3.20
2.80
2.40
2.00
2.50
3.00
3.50
4.00
4.50
5.00
K VALUE

Figure II.E.2

over the other with respect to precision or accuracy. However, the method of moments is considerably cheaper with respect to computation time required. This is consistent with known results that for i.i.d. Gamma data ($k = q$), the moment estimate for k is quite efficient when compared to the maximum likelihood estimator.

Third case ($k = 0.75$, $q = 0.1825$). The third type of data was a high correlation case with a low k value. Specifically $k = 0.75$, $q = 0.1825$, and the correlation was 0.75. As can be seen in Figure II.E.3 and Table II.E.4, both the methods of parameter estimation considerably overestimated the parameter values, indicating considerable bias in the procedures. The method of moments produced estimates for k with sample mean of 0.8061 and sample standard deviation of 0.067. Those for q had a sample mean of 0.232 and sample standard deviation of 0.026. The likelihood method produced estimates for k with a sample mean of 0.852 and sample standard deviation 0.039. The corresponding statistics for q estimates are 0.224 and 0.004. As was true in the other high correlation case, the standard deviations of the maximum likelihood estimates are considerably smaller than those of the moment estimates. However, since the evidence is that the estimates are highly biased, the advantage of this smaller standard deviation is not clear unless additional data would serve to reduce the apparent bias. The detailed results for this data type are presented in Table II.E.4 and Figure II.E.3. It would be of interest

TABLE II.E.4

Results of Search for Maximum Likelihood Estimates in a GLAR(1) Model; $k = 0.75$, $q = 0.1875$

Data Set	Starting Value k	Starting Value q	Run Time (in Min.)	Number of Iterations	Ending Value k	Ending Value q	Maximum Likelihood	Value of Difference (10^{-3})
0	0.7500	0.1825	85	5	0.861	0.221	325.685	2
0	0.9795	0.2489	136	4	0.863	0.222	325.685	
1	0.7500	0.1825	59	4	0.832	0.235	866.216	0
1	0.8026	0.2351	19	1	0.832	0.235	866.214	
2	0.7500	0.1825	72	4	0.806	0.224	439.198	1
2	0.7621	0.2147	66	4	0.807	0.224	439.198	
3	0.7500	0.1825	141	2	0.866	0.222	1131.684	0
3	0.7514	0.2213	121	1	0.866	0.222	1131.671	
4	0.7500	0.1825	59	4	0.807	0.223	2333.130	1
4	0.7749	0.2420	54	5	0.807	0.222	2333.131	
5	0.7500	0.1825	59	4	0.910	0.217	883.801	0
5	0.8336	0.2320	210	4	0.910	0.217	883.801	
6	0.7500	0.1825	129	5	0.885	0.222	287.798	1
6	0.7308	0.1790	88	4	0.884	0.222	287.799	
7	0.2500	0.1825	71	5	0.795	0.223	977.265	0
7	0.7759	0.2241	117	8	0.795	0.223	977.265	
8	0.7500	0.1825	66	6	0.906	0.227	1018.470	0
8	0.8092	0.2863	136	6	0.906	0.227	1018.472	
9	0.7500	0.1825	57	4	0.818	0.212	410.842	36
9	0.8335	0.2347	85	3	0.852	0.225	411.703	

AD-A115 754 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 12/1
EXTENSION OF SOME MODELS FOR POSITIVE-VALUED TIME SERIES.(U)
MAR 82 D K HUGUS

UNCLASSIFIED NL

2 OF 4
AD A
115754

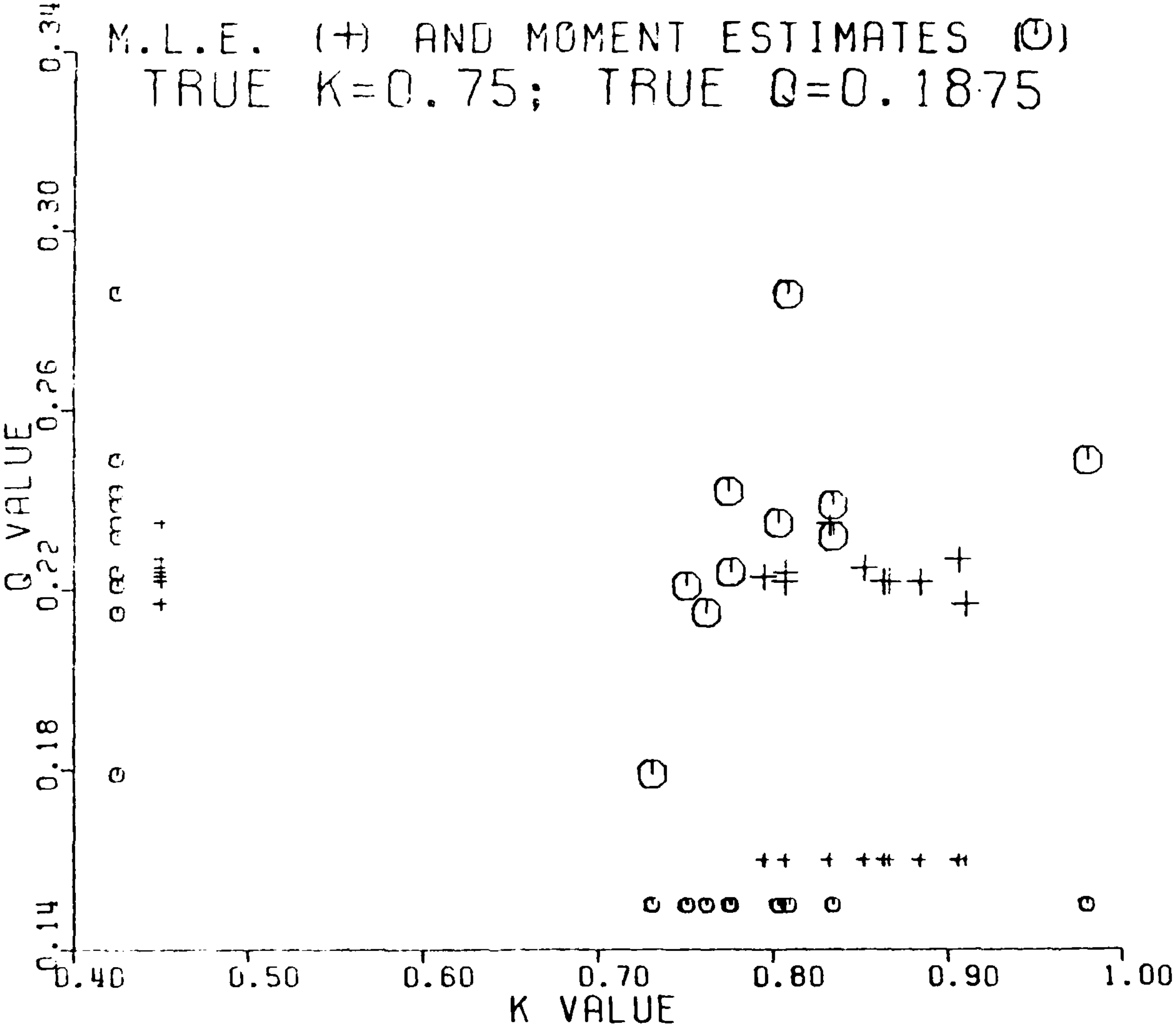

M.L.E. (+) AND MOMENT ESTIMATES (O)
TRUE K=0.75; TRUE Q=0.1875
Q VALUE
0.34
0.30
0.26
0.22
0.18
0.14
0.40
0.50
0.60
0.70
0.80
0.90
1.00
K VALUE

Figure II.E.3

to see if a technique such as (2-fold) jacknifing, such as that applied by Quenouille [Ref. 17] to correlation estimates, would help here.

A much larger study would be needed to come to definite conclusions about the efficiency of maximum likelihood estimation in this model. However, as in the i.i.d. case for Gamma variates, m.l.e.'s are likely to be considerably better than moment estimations for values of k less than one.

Note too that the maximum likelihood estimation did not include the mean value parameter. This could be done or the mean could be estimated from the sample mean $\bar{X}$. The inflation of variation of $\bar{X}$ due to the correlation is known to be (asymptotically)

$$1 + 2 \sum_{k=1}^{\infty} \rho^k = \frac{1 + \rho}{1 - \rho}$$

Thus for $\rho = 0.75$, the effective sample size for estimating μ in a sample size of size n is $n(1-\rho)/(1+\rho)$. For $\rho = 0.75$ this is $n/7$.

The normality of the estimates was investigated with normal plots and Shapiro-Wilk tests for normality. A summary of results if given in Table II.E.5. The normality hypothesis is accepted at a 0.95 level if the Shapiro-Wilk statistic W is higher than 0.842, at a 0.99 level level if it is higher than 0.781. No strong indication of non-normality is indicated in any case.

Table II.E.5

Summary of Simulation Results

Case	Estimate	Estimated Mean	Estimated Std. Dev.	Shapiro-Wilks	Accept 5%	Accept 1%
Case I. $k = 4.0$ $q = 1$	$\hat{k}$	3.93	0.27	.874	Yes	Yes
	$\tilde{k}$	3.94	0.55	.918	Yes	Yes
	$\hat{q}$	1.00	0.11	.914	Yes	Yes
	$\tilde{q}$	0.96	0.21	.941	Yes	Yes
Case II. $k = 4.0$ $q = 1$	$\hat{k}$	4.00	0.16	.963	Yes	Yes
	$\tilde{k}$	3.99	0.17	.933	Yes	Yes
	$\hat{q}$	2.99	0.17	.870	Yes	Yes
	$\tilde{q}$	2.98	0.18	.970	No	Yes
Case III. $k = 0.75$ $q = 0.1875$	$\hat{k}$	0.852	0.039	.935	Yes	Yes
	$\tilde{k}$	0.806	0.067	.822	No	Yes
	$\hat{q}$	0.224	0.004	.846	Yes	Yes
	$\tilde{q}$	0.232	0.026	.940	Yes	Yes

III. MOVING AVERAGE MODELS

A. INTRODUCTION

Although several researchers have proposed models for correlated, marginally exponential random sequences [Refs. 18, 19, 20], Gaver and Lewis [Ref. 2] produced the first analytically and computationally tractable model for the generation of correlated, marginally Exponential random sequences. They showed that in the usual linear, additive, first-order, autoregressive equation

$$X_n = \beta X_{n+1} + \varepsilon_n \qquad \text{(III.A.1)}$$

where $\{X_n, n = 0,1,2,\ldots\}$ is a second-order stationary, marginally Exponentially distributed sequence of random variables; $\{\varepsilon_n, n = 1,2,\ldots\}$ is an independent, identically distributed (iid) sequence of innovative random variables; $0 \leq \beta < 1$, the distribution of the $\{\varepsilon_n\}$ which produces the desired marginal distribution for $\{X_n\}$ is

$$\varepsilon_n = \begin{cases} 0 & \text{with probability } \beta, \\ E_n & \text{with probability } 1-\beta, \end{cases} \qquad \text{(III.A.2)}$$

where $\{E_n, n = 1,2,\ldots\}$ is an iid sequence of Exponentially distributed random variables with the same parameter, λ, as the $\{X_n\}$ sequence. Equation III.A.1 can now be written as

$$X_n = \begin{cases} \beta X_{n-1} & \text{with probability } \beta, \\ \beta X_{n-1} + E_n & \text{with probability } 1-\beta. \end{cases} \qquad \text{(III.A.3)}$$

If $\{I_n, n = 1,2,...\}$ is an iid binary sequence independent of $\{X_n\}$ and $\{E_n\}$ such that $P(I_n = 0) = 1 - P(I_n = 1) = \beta$, then equation III.A.3 can be more succinctly written as

$$X_n = \beta X_{n-1} + I_n E_n. \qquad \text{(III.A.4)}$$

X_n is a random linear combination of identically distributed random variables in the sense that the variable E_n actually enters into the sum only when the random variable I_n has value one. Since for a given $\{I_n\}$ sequence, the distribution of X_n depends only on the distribution of X_{n-1} and E_n, X_n will be Exponentially distributed whenever both X_{n-1} and E_n are independent and have the same Exponential distribution. This understanding allows the autoregressive relation in III.A.4 to be transformed into a moving average by substituting another innovative random variable for the X_{n-1} to produce

$$X_n = \beta E_n + I_n E_{n+1}. \qquad \text{(III.A.5)}$$

This model was designated the EMA(1) for Exponential moving average of order one. This EMA(1) model is one dependent in that X_n and X_{n+j} are independent for $j \neq \pm 1$. Consequently, only the lag one correlation, $\rho_1 = \rho_{-1} = \text{CORR}(X_n, X_{n+1})$, or more completely only the joint distribution of X_n and X_{n+1} need be studied.

Lawrance and Lewis [Ref. 5] give a fairly complete description of this EMA(1) model. Of particular note is the relative tractability of this model which enabled the authors to derive correlations, distributions of sums of X_n's, intensity function, spectrum of counts, joint density of X_n and X_{n+1}, conditional expectations, and other properties. The existence of these characteristics is beneficial in data analysis and is a primary advantage of the EMA(1) over previously suggested models. However, the EMA(1) model does possess a limitation. The range of possible positive correlations, ρ_1, is restricted to the interval from zero to one-quarter. Thus for a given correlation between zero and one-quarter, the structure of X_n and X_{n+1} and the sample path behavior of the sequence are determined.

The structure of III.A.1 is that of a special random linear combination of Exponential random variables to given an Exponential random variable. Other such random linear combinations are now known and for the first-order autoregressive case produce dramatic differences in sample path behavior of the sequence $\{X_n\}$. In this section of the thesis we investigate these random linear combinations in the context of the first-order moving average structure.

In this Chapter we examine extensions of the model in four ways:

1. <u>Negative Correlation</u>

McKenzie [Ref. 21] has suggested a scheme for inducing negative correlation in the EMA(1) process by correlating the

sequence $\{I_n\}$. A better scheme, in that it produces a larger range of negative correlations, was introduced by Lawrance and Lewis [Ref. 8] for the extended first-order autoregressive model NEAR(1). This scheme involves a bivariate error sequence $\{E_n, E_n'\}$ and its use is investigated in this thesis for the moving average process.

2. New Exponential Moving Average Model of Order One, NEMA(1)

It will be shown that no first-order moving average process which is a random linear combination of Exponential random variables can have correlation greater than one-quarter. Thus the differences in the processes for given correlation is investigated in terms of the joint structure of X_n and X_{n+1}. The NEMA(1) process obtained by using the NEAR(1) structure [Ref. 8] in a moving average context is analytically tractable and quantities such as the joint Laplace-Stiltjes transform of X_n and X_{n+1}, the spectrum of counts, $P(X_{n+1} > X_n)$, and conditional expectations are obtained. It also combines the forward and backward EMA(1) models as extreme cases and is thus a natural extension of the EMA(1) model.

3. The Moving Minimum Model

A non-linear combination of Exponentials involving minima has been applied by Tavares [Ref. 22] to obtain a first-order autoregressive process which is intimately connected [Ref. 23] with the EAR(1) process of Gaver and Lewis [Ref. 24]. This structure is applied to the moving average process. It is shown that this process extends the range of attainable

correlations in first-order "moving average" process beyond that obtainable by random linear combinations of Exponentials. However, the process is slightly degenerate in that there is a positive probability that successive points will lie on the line $X_{n+1} = bX_n$ for positive correlations and on the curve $e^{-X_n} + e^{-X_{n+1}} = 1$ for negative correlations. The price paid for the extended range of correlations is a limited analytical tractability as compared to the EMA(1) or NEMA(1) processes. The moving minimum process is investigated in terms of the joint structure of X_n and X_{n+1}. Although the joint distribution can be derived, the functions are difficult to examine in detail. Therefore, simple characterizations of the joint structure, in addition to the correlation, are examined. These include the $P(X_{n+1} > X_n)$, a crude measure of the tendency of the sequence to exhibit runs up and down, and conditional expectations, $E(X_n | X_{n-1} = x)$ and $E(X_{n-1} | X_n = y)$.

4. The Beta-Exponential Model

Finally, another random linear combination of Exponentials to produce correlated Exponentials is examined. Unlike the previous models, the coefficients of the Exponential random variables are themselves continuous random variables. This increases the complexity and reduces the analytical tractability of this model. Simple sample path characteristics are derived or simulated. These are special cases of the GLMA(1) from Chapter II.

B. NEGATIVE CORRELATION IN MOVING AVERAGE MODELS

The problem of non-negative correlations was addressed by McKinzie [Ref. 21] who modified the form of the EMA(1) model to be:

$$X_n = \beta E_n + (1 - Z_n)E_{n-1}, \qquad \text{(III.B.1)}$$

where $\{E_n, n = 0,1,2,\ldots\}$ is an iid sequence of Exponential random variables, $\{Z_n, n = 1,2,\ldots\}$ is a sequence of binary random variables with $P(Z_n = 1) = 1 - P(Z_n = 0) = \beta$, and Z_n is independent of all E_n and all past X_n. By imposing an MA(1) correlation on the sequence $\{Z_n\}$, McKenzie was able to produce a negative correlation for the $\{X_n\}$. However, this negative correlation is achieved at the cost of reducing the possible range of positive correlations for the $\{X_n\}$. Using McKenzie's formulation, the range of correlations obtainable with the EMA(1) model is $(-\frac{1}{64},\frac{1}{16})$.

An alternative procedure for producing negative correlations was introduced by Lawrance and Lewis [Ref. 8]. Their procedure requires two series of innovative factors $\{E_n, n = 0,1,\ldots\}$ and $\{E_n', n = 0,1,\ldots\}$ which are correlated and may be antithetic.

Antithetic variables are generated by using the fact that the variables E_i and E_i' defined as:

$$\begin{aligned} E_i &= -\ln(U_i), \\ E_i' &= -\ln(1-U_i), \end{aligned} \qquad \text{(III.B.2)}$$

where $\{U_i, i = 1,2,\ldots\}$ is a sequence of uniform (0,1) random variables, are both marginally Exponentially distributed and correlated. P.A.P. Moran [Ref. 25] has determined that the

correlation between E_i and E_i' is approximately -0.6449 and this is the lowest correlation possible between Exponential random variables. E_i and E_i' have a deterministic relationship since

$$E_i' = -\ln(1 - e^{-E_i}). \qquad \text{(III.B.3)}$$

Using the Lawrance and Lewis extension of the EMA(1) process, the model becomes

$$X_n = \beta E_n + I_n E'_{n-1}, \qquad \text{(III.B.4)}$$

where $\{E_n, n = 1,2,...\}$ is an iid sequence of Exponential random variables, $\{E_n', n = 0,1,...\}$ is a sequence of Exponential random variables which are correlated with the respective variables in the $\{E_n\}$ sequence, $\{I_n, n = 1,2,...\}$ is a sequence of independent binary random variables with $P(I_n = 0) = 1 - P(I_n = 1) = \beta$, $0 \le \beta \le 1$, and $\{I_n\}$, $\{E_n\}$ are independent of each other and all previous X_n values.

The correlation of the X's can then be computed as follows. Let $E(X) = \mu$ and recall that since $\{X_n\}$ and $\{E_n\}$ have the same marginal exponential distributions, $VAR(X_n) = VAR(E_n)$.

$$\begin{aligned} X_{n+1}X_n &= [\beta E_{n+1} + I_{n+1}E_n'][\beta E_n + I_n E'_{n-1}] \\ &= \beta^2 E_{n+1}E_n + \beta I_{n+1}E_n E_n' + \beta I_n E_{n+1}E'_{n-1} + I_{n+1}I_n E_n' E'_{n-1}. \end{aligned}$$

Thus, using the independence of $\{E_n\}$, $\{I_n\}$, and the iid nature of $\{E_n\}$ and $\{I_n\}$

$$E(X_{n+1}X_n) = \beta^2\mu^2+\beta(1-\beta)[COV(E_n,E_n')+\mu^2]+\beta(1-\beta)\mu^2+(1-\beta)^2\mu^2$$

$$= \mu^2 + \beta(1-\beta)COV(E_n,E_n')$$

Therefore,

$$COV(X_{n+1},X_n) = \beta(1-\beta)COV(E_n,E_n')$$

and using VAR(X) = VAR(E)

$$CORR(X_{n+1},X_n) = \beta(1-\beta)CORR(E_n,E_n') \qquad (III.B.5)$$

Using Moran's result [Ref. 25] the correlation of antithetic Exponentials is known to be approximately -0.6449. Therefore, the greatest negative value that can be achieved for $CORR(X_{n+1},X_n)$ is approximately -0.1612 when $\beta = 0.5$. Since no restriction was placed on $CORR(E_n,E_n')$, the sequences $\{E_n\}$ and $\{E_n'\}$ need not be antithetic, but can have any correlation that is possible for two Exponential sequences with the same marginal distribution. By specifying that $E_n' = E_n$, the original EMA(1) is achieved and the correlation for the X's is $\beta(1-\beta)$ as in Lawrance and Lewis [Ref. 5]. By allowing the correlation between the $\{E_n\}$ and $\{E_n'\}$ sequences to vary from -0.6449 to 1.0, the correlation of the X's will vary from a minimum of

-0.1612 to a maximum of 0.25 (also depending on the value of β) as can be seen from III.B.5. The Lawrance and Lewis extension of the EMA(1) model gives greater possible variation in the correlation than that of McKenzie, but requires two sequences of Exponential random variables to achieve this range.

Although it is clear that with $E_n' = E_n$, a $CORR(E_n, E_n') = 1$ and when $\{E_n\}$ and $\{E_n'\}$ are antithetic $CORR(E_n, E_n') = -0.6449$, it may not be obvious how to generate $\{E_n'\}$ sequences with correlations between these two extreme values. A simple bivariate exponential sequence with any desired correlation in the permissible range can be generated using the relationship

$$E_i' = \begin{cases} E_i & \text{with probability } p, \\ \bar{E}_i & \text{with probability } 1-p, \end{cases} \qquad \text{(III.B.6)}$$

where $\bar{E}_i$ is the antithetic of E_i. Then the extended EMA(1) model has two parameters, β and p, and the range of correlations for the X's is -0.1612 to 0.25, as above. The bivariate density for the pair $\{E_i', E_i\}$ is not smooth. Other bivariate densities such as those in Gaver [Ref. 26] and Lawrance and Lewis [Ref. 27] can also be used.

The above ideas on extending the correlation structure of the moving average models to encompass negative correlation can be applied to all of the new models given below. Details are not given.

C. THE EXTENDED EMA(1) MODEL, NEMA(1)

1. Introduction

The original Exponential moving average process is discussed by Lawrance and Lewis [Ref. 5]. This paper considers the first order process defined by:

$$X_n = \begin{cases} \beta E_n & \text{with probability } \beta, \\ \beta E_n + E_{n+1} & \text{with probability } (1-\beta), \end{cases} \qquad \text{(III.C.1)}$$

where $\{E_n, n = 0, \pm 1, \pm 2, \ldots\}$ is an iid Exponential sequence and $0 \leq \beta \leq 1$. This random linear combination of Exponential variates is called the EMA(1) model for Exponential moving average of order one. Since X_n is a function of E_n and E_{n+1}, this version is called the forward EMA(1). The backward version of EMA(1) is obtained when E_{n+1} is replaced by E_{n-1} in III.C.1.

The fact that EMA(1) is a single parameter model suggests that this model may not be sufficiently flexible to address all processes. Investigation of an alternate, two parameter model may indicate that a two parameter model is sufficiently more flexible to justify its increased complexity.

The extended, two parameter model is based on the new Exponential autoregressive process of order one (NEAR(1)). The NEAR(1) model propounded by Lawrance and Lewis [Ref. 8] is defined as

$$X_n = \begin{cases} \mathcal{E}_n + \beta X_{n-1} & \text{with probability } \alpha \\ \mathcal{E}_n & \text{with probability } (1-\alpha) \end{cases} \qquad \text{(III.C.1.2)}$$

where $\{X_n, n = 1,2,\ldots\}$ is a second-order stationary sequence of Exponential random variables with parameter λ, $\{\mathcal{E}_n\}$ is an iid sequence of innovative factors, $0 \leq \beta \leq 1$, $0 \leq \alpha \leq 1$, and $\alpha+\beta < 2$. By letting $\phi_X(s)$ and $\phi_\varepsilon(s)$ denote the Laplace-Stieltjis transform of X and $\mathcal{E}$ respectively, Lawrance and Lewis determined that $\phi_\varepsilon(s) = \frac{\lambda+\beta s}{\lambda+s} \cdot \frac{\lambda}{\lambda+(1-\alpha)\beta s}$. Using a partial fraction solution technique to invert this transform produced

$$\mathcal{E}_n = \begin{cases} E_n & \text{with probability } \frac{1-\beta}{1-(1-\alpha)\beta} \\ (1-\alpha)\beta E_n & \text{with probability } \frac{\alpha\beta}{1-(1-\alpha)\beta}. \end{cases} \qquad \text{(III.C.1.3)}$$

where $\{E_n, n = 0,1,2,\ldots\}$ is an iid sequence of Exponentially distributed random variables which has the same parameter as the $\{X_n\}$ sequence.

By noticing that the autoregressive model given in III.C.1.2 using III.C.1.3 is a random sum of two iid Exponentially distributed random variables, the NEMA(1) model is produced by substituting E_{n-1} for X_{n-1} in the NEAR(1) model. This procedure is identical to that used to produce the EMA(1) model from the EAR(1) model and yields

$$X_n = \begin{cases} \mathcal{E}_n + \beta E_{n-1} & \text{with probability } \alpha, \\ \mathcal{E}_n & \text{with probability } 1-\alpha. \end{cases} \qquad \text{(III.C.1.4)}$$

This model can be written in a more compact form as

$$X_n = K_n E_n + I_n E_{n-1}, \tag{III.C.1.5}$$

where $\{X_n, n = 1,2,...\}$ is a second order stationary sequence of marginally Exponentially distributed random variables; $\{E_n, n = 0,1,...\}$ is an iid sequence of Exponential random variables with the same parameter as the $\{X_n\}$ sequence; $\{I_n, n = 1,2,...\}$ is an iid sequence of random variables with $P(I_n = \beta) = 1-P(I_n = 0) = \alpha$; $\{K_n, n = 1,2,...\}$ is an iid sequence of random variables with $P(K_n = 1) = 1-P(K_n = (1-\alpha)\beta) = \frac{1-\beta}{1-(1-\alpha)\beta}$; $\{I_n\}$, $\{K_n\}$, and $\{E_n\}$ are mutually independent; $0 \leq \alpha \leq 1$; $0 \leq \beta \leq 1$; and $\alpha+\beta < 2$.

The NEMA(1) model contains both the forward and backward versions of EMA(1) as special cases. When $\alpha = 1$; $P(I_n = \beta) = 1$, $(1-\alpha)\beta = 0$, and $P(K_n = 0) = \beta$. Hence, the NEMA(1) model becomes

$$X_n = \begin{cases} \beta E_{n-1} & \text{with probability } \beta, \\ \beta E_{n-1} + E_n & \text{with probability } (1-\beta). \end{cases} \tag{III.C.1.6}$$

This is a form of the forward EMA(1).

When $\beta = 1$; $P(I_n = 1) = \alpha$, $(1-\alpha)\beta = (1-\alpha)$, and $P(K_n = (1-\alpha)) = \frac{\alpha\beta}{1-(1-\alpha)\beta} = 1$. In this case, the NEMA(1) model becomes

$$X_n = \begin{cases} (1-\alpha)E_n & \text{with probability } (1-\alpha), \\ (1-\alpha)E_n + E_{n-1} & \text{with probability } \alpha. \end{cases} \qquad \text{(III.C.1.7)}$$

This is a form of the backward EMA(1) with $\beta = 1-\alpha$. Therefore, the NEMA(1) contains the special cases of EMA(1) when α and β assume specific values.

One should also note in passing that the $\{X_n\}$ sequence becomes an iid sequence if $\alpha = 0$ or $\beta = 0$.

2. <u>Correlation Structure</u>

The following relationships will prove of value in succeeding calculations

$$P(I_n = \beta) = 1 - P(I_n = 0) = \alpha. \qquad \text{(III.C.2.1)}$$

$$E(I_n) = \alpha\beta + (1-\alpha)\cdot 0 = \alpha\beta. \qquad \text{(III.C.2.2)}$$

$$\begin{aligned} P(K_n = 1) &= 1-P(K_n = (1-\alpha)\beta) \qquad \text{(III.C.2.3)} \\ &= \frac{1-\beta}{1-(1-\alpha)\beta}. \end{aligned}$$

$$\begin{aligned} E(K_n) &= 1\left(\frac{1-\beta}{1-(1-\alpha)\beta}\right) + (1-\alpha)\beta\left(\frac{\alpha\beta}{1-(1-\alpha)\beta}\right) \\ &= \frac{1-\beta+\alpha\beta^2-\alpha^2\beta^2}{1-\beta+\alpha\beta} \\ &= \frac{1-\beta+\alpha\beta-\alpha\beta+\alpha\beta^2-\alpha^2\beta^2}{1-\beta+\alpha\beta} \\ &= \frac{1-\beta+\alpha\beta}{1-\beta+\alpha\beta} - \frac{\alpha\beta(1-\beta+\alpha\beta)}{1-\beta+\alpha\beta} \end{aligned}$$

$$E(K_n) = 1-\alpha\beta \tag{III.C.2.4}$$

$$X_n = K_nE_n + I_nE_{n-1} \tag{III.C.2.5}$$

$$\begin{aligned} E(X_n) &= E(K_nE_n + I_nE_{n-1}) \\ &= E(K_nE_n) + E(I_nE_{n-1}) \\ &= E(K_n)E(E_n) + E(I_n)E(E_{n-1}) \\ &= (1-\alpha\beta)E(E_n) + \alpha\beta E(E_{n-1}) \end{aligned} \tag{III.C.2.6}$$

$$E(X) = E(E)$$

Since $\{X_n\}$ and $\{E_n\}$ are both Exponential, $E(X) = E(E)$ implies $VAR(X) = VAR(E)$. Since both $\{E_n\}$ and $\{X_n\}$ have the same Exponential parameter, without loss of generality this parameter will be considered to be 1. This, of course, requires $E(X) = 1$ and $VAR(X) = 1$.

The possible range of correlations for the NEMA(1) model can be determined by a simple calculation. We have

$$X_n = K_nE_n + I_nE_{n-1},$$

$$X_{n+1} = K_{n+1}E_{n+1} + I_{n+1}E_n.$$

Thus

$$X_n X_{n+1} = (K_n E_n + I_n E_{n-1})(K_{n+1}E_{n+1} + I_{n+1}E_n)$$

$$= K_{n+1}K_n E_{n+1}E_n + K_{n+1}I_n E_{n+1}E_{n-1} + K_n I_{n+1}E_n^2$$

$$+ I_{n+1}I_n E_n E_{n-1}$$

Using the independence of $\{K_n\}$, $\{I_n\}$ and $\{E_n\}$ and the iid nature of these three sequences, we have

$$E(X_n X_{n+1}) = (1-\alpha\beta)^2[E(E_n)]^2 + (1-\alpha\beta)\alpha\beta[E(E_n)]^2$$

$$+ (1-\alpha\beta)\alpha\beta[2\{E(E_n)\}^2] + (\alpha\beta)^2[E(E_n)]^2$$

$$= 1 + (1-\alpha\beta)\alpha\beta$$

Therefore,

$$COV(X_n, X_{n+1}) = (1-\alpha\beta)\alpha\beta$$

and

$$CORR(X_n, X_{n+1}) = (1-\alpha\beta)\alpha\beta. \qquad \text{(III.C.2.7)}$$

Therefore, the original NEMA(1) model has the same range of possible correlations as the EMA(1), namely that the correlations must lie in the interval $[0,\frac{1}{4}]$. One should note that it is not possible to distinguish the parameters α and β from the correlation, or even whether the product, $\alpha\beta$, has a given value or one minus that value. This is similar to the normal moving

average model of order one where the cases ϕ and $1/\phi$ are indistinguishable. In the normal case, the range of ϕ is limited to the interval [0,1] on the basis of invertibility [Ref. 28]. It would seem simple and convenient here to limit $\alpha\beta$ to the interval $[0,\frac{1}{2}]$. However, non-normal processes are not completely determined by their correlation structure. In fact, Jacobs and Lewis [Ref. 6] showed that in the EMA(1) case, the values β and $(1-\beta)$ can be distinguished using directional moments, $E(X_n X_{n+1}^2)$ and $E(X_n^2 X_{n+1})$. Hence, such a restriction on the value of $\alpha\beta$ is inappropriate.

One should also note that, since the correlation between X_n and X_{n+K} is zero for all K with absolute value greater than one, the first-order correlation completely determines the correlation structure of the model.

The restriction on the range of attainable correlation is disappointing but not surprising since it can be proven that any Exponential moving average process of order one generated as a linear combination of independent Exponentials must have a correlation that lies in the range $[0,\frac{1}{4}]$. The proof of this contention follows the previous calculation closely.

THEOREM:

Assume $\{E_n, n = 0,1,2,\ldots\}$ is a sequence of iid Exponential variables with unit mean, $\{A_n, n = 1,2,\ldots\}$ and $\{B_n, n = 1,2,\ldots\}$ are sequences of iid random variables, and $\{A_n\}$, $\{B_n\}$, $\{E_n\}$ are all mutually independent. Moreover, assume the sequence $\{X_n, n = 1,2,\ldots\}$ defined by

$$X_n = A_n E_n + B_n E_{n-1} \quad \text{(III.C.2.8)}$$

is a unit mean Exponential sequence. Now

$$E(X_n) = E(A_n E_n + B_n E_{n-1})$$

$$= E(A_n)E(E_n) + E(B_n)E(E_{n-1})$$

$$1 = E(A_n) + E(B_n) \quad \text{(III.C.2.9)}$$

In addition, since $X_n > 0$ for all n, both A_n and B_n must be non-negative for all n. Hence $E(A_n) \geq 0$ and $E(B_n) \geq 0$. It also follows that $1 \geq E(A)$ and $1 \geq E(B)$. Now

$$X_n X_{n+1} = (A_n E_n + B_n E_{n-1})(A_{n+1}E_{n+1} + B_{n+1}E_n)$$

Therefore

$$E(X_n X_{n+1}) = E(A_{n+1}A_n E_{n+1}E_n + A_{n+1}B_n E_{n+1}E_{n-1} + A_n B_{n+1} E_n^2$$

$$+ B_{n+1}B_n E_n E_{n-1})$$

$$= [E(A)]^2 + E(A)E(B) + 2E(A)E(B) + [E(B)]^2$$

$$= [E(A) + E(B)]^2 + E(A)E(B)$$

Since $E(A) + E(B) = 1$ from III.C.2.9, $E(A) + E(B) = 1$.

$$E(X_n X_{n+1}) = 1 + E(A)E(B)$$

Therefore, since $E(X_n)$ is one, by assumption,

$$\begin{aligned} COV(X_n, X_{n+1}) &= E(A)E(B) \\ &= E(A)[1 - E(A)] \end{aligned}$$

and

$$CORR(X_n, X_{n+1}) = E(A)[1 - E(A)] \qquad \text{(III.C.2.10)}$$

Since $0 \leq E(A) \leq 1$, the correlation must lie in the interval $[0, \frac{1}{4}]$. Q.E.D.

The possible range of correlations can be extended by reformating the model. We choose to do this first by using the method devised by Lawrance and Lewis [Ref. 8] and given in equation III.B.4. With variables and sequences defined as in equation III.C.1.5 let $\{E_n\}$ and $\{E_n'\}$ be correlated sequences and redefine the NEMA(1) as

$$X_n = K_n E_n + I_n E_{n-1}'. \qquad \text{(III.C.2.11)}$$

Then it follows that

$$\begin{aligned} X_n X_{n+1} &= (K_n E_n + I_n E_{n-1}')(K_{n+1} E_{n+1} + I_{n+1} E_n') \\ &= K_{n+1} K_n E_{n+1} E_n + K_{n+1} I_n E_{n+1} E_{n-1}' + K_n I_{n+1} E_n E_n' \\ &\quad + I_{n+1} I_n E_n' E_{n-1}' \end{aligned}$$

Thus

$$E(X_n X_{n+1}) = (1-\alpha\beta)^2 + (1-\alpha\beta)\alpha\beta + (1-\alpha\beta)\alpha\beta[\mathrm{COV}[E_n,E_n']+1] + \alpha^2\beta^2$$

$$= 1 + (1-\alpha\beta)\alpha\beta\,\mathrm{COV}(E_n,E_n')$$

and

$$\mathrm{COV}(X_n,X_{n+1}) = (1-\alpha\beta)\alpha\beta\,\mathrm{COV}(E_n,E_n')$$

Fina ly

$$\mathrm{CORR}(X_n,X_{n+1}) = (1-\alpha\beta)\alpha\beta\,\mathrm{CORR}(E_n,E_n'). \qquad \text{(IIIC.2.12)}$$

As described above, Moran [Ref. 25] has determined that the range of possible correlations for two Exponentials is (-0.6449,1.0). Thus when $\alpha\beta = 0.5$, the possible correlations for X_n and X_{n+1} fall in the interval (-0.1612,0.25). This procedure extends the range of possible correlations at the cost of generating the additional $\{E_n'\}$ sequence.

McKenzie [Ref. 21] has suggested that the range of correlations could be extended by requiring that the $\{I_n\}$ sequence be correlated. Using this scheme, he was able to show that the correlations for the $\{X_n\}$ sequence lies in the interval $(-\frac{1}{64},\frac{1}{16})$. Because of the requirement of the moving average process of order one to have zero correlation for lags greater than one, the correlation of the $\{I_n\}$ sequence also

had to be of MA(1) type. It is this restriction that produces such a narrow range of correlations. A logical and obvious extension to McKenzie's work is to require that both the $\{K_n\}$ and $\{I_n\}$ sequences in the NEMA(1) model have a MA(1) correlation structure. This can be combined with the correlated $\{E_n\}$, $\{E_n'\}$ scheme of Lawrance and Lewis. If this combined case is carried out, the NEMA(1) model is as follows

$$X_n = K_n E_n + I_n E_{n-1}', \tag{III.C.2.13}$$

where $\{K_n, n = 1,2,...\}$ is a sequence of random variables with an MA(1) correlation structure with $P(K_n = 1) = 1-P(K_n = (1-\alpha)\beta) = \frac{1-\beta}{1-(1-\alpha)\beta}$; $\{I_n, n = 1,2,...\}$ is a sequence of random variables with an MA(1) correlation structure with $P(I_n = \beta) = 1-P(I_n = 0) = \alpha$; $\{E_n\}$ and $\{E_n'\}$ are correlated sequences with marginal Exponential distributions with unit means; and $\{K_n\}$, $\{I_n\}$, and $\{E_n\}$ are mutually independent.

Now

$$\begin{aligned} X_n X_{n+1} &= (K_n E_n + I_n E_{n-1}')(K_{n+1} E_{n+1} + I_{n+1} E_n') \\ &= K_{n+1} K_n E_{n+1} E_n + K_{n+1} I_n E_{n+1} E_{n-1}' + K_n I_{n+1} E_n E_n' \\ &\quad + I_{n+1} I_n E_n' E_{n-1}' \end{aligned}$$

So

$$E(X_nX_{n+1}) = E(K_{n+1}K_n)+(1-\alpha\beta)\alpha\beta+(1-\alpha\beta)\alpha\beta E(E_nE_n')+E(I_{n+1}I_n)$$

$$= COV(K_{n+1},K_n)+(1-\alpha\beta)^2+(1-\alpha\beta)\alpha\beta$$

$$+(1-\alpha\beta)\alpha\beta\cdot[COV(E_n,E_n')+1]+COV(I_{n+1},I_n)+(\alpha\beta)^2$$

$$= 1+COV(K_{n+1},K_n)+COV(I_{n+1},I_n)+(1-\alpha\beta)\alpha\beta COV(E_n,E_n')$$

Therefore

$$COV(X_n,X_{n+1}) = COV(K_{n+1},K_n)+COV(I_{n+1},I_n)+(1-\alpha\beta)\alpha\beta COV(E_n,E_n')$$

and

$$CORR(X_n,X_{n+1}) = COV(K_{n+1},K_n)+COV(I_{n+1},I_n) +(1-\alpha\beta)\alpha\beta COV(E_n,E_n') \quad \text{(III.C.2.14)}$$

Although this scheme obviously extends the range of possible correlations, it does so at the expense of considerable complexity. Considering the limited range of correlations possible by imposing a correlation on the $\{I_n\}$ and $\{K_n\}$ sequences, the additional complexity may not be warranted. If in spite of the complexities involved, one decides to induce correlations in the coefficient sequences, the NEMA(1) because it has two such sequences will yield a slightly larger range than the EMA(1) model.

3. $\underline{P(X_{n+1} > X_n)}$

One of the possible advantages of a two parameter model is the capacity for modifying $P(X_{n+1} > X_n)$ and, consequently, the sample path behavior of the process while maintaining a constant correlation. Since the correlation is a function of $\alpha\beta$, one can vary the values of α and β while keeping the product constant. The $P(X_{n+1} > X_n)$ can be calculated by addressing each of the sixteen possible combinations of K and I values for X_{n+1} and X_n, computing the probability for each combination, and weighting the probability associated with a given combination by the probability that the given combination occurs. A sample calculation is provided and complete results presented in Table III.C.3.1.

Example: We have

$$X_n = K_n E_n + I_n E_{n-1},$$

$$X_{n+1} = K_{n+1} E_{n+1} + I_{n+1} E_n,$$

$$P(K_n = 1) = 1-P(K_n = (1-\alpha)\beta) = \frac{1-\beta}{1-(1-\alpha)\beta},$$

$$P(I_n = \beta) = 1-P(I_n = 0) = \alpha.$$

Let $\frac{1-\beta}{1-(1-\alpha)\beta} = \delta$ and consider the case where

$$I_n = \beta,\ K_n = 1,\ I_{n+1} = \beta,\ K_{n+1} = 1.$$

Since the $\{I_n\}$ and $\{K_n\}$ sequences are both iid and independent of each other, the probability of this combination of parameter values is simply the product of the individual probabilities of occurrence. Hence the probability of occurrence is $\alpha^2\delta^2$. Then in this case

$$
\begin{aligned}
P(X_{n+1} > X_n) &= P(E_{n+1}+\beta E_n > E_n+\beta E_{n-1}) \\
&= P(E_{n+1} > (1-\beta)E_n+\beta E_{n-1}) \qquad \text{(III.C.3.1)}
\end{aligned}
$$

Now E_{n+1} is independent of $Y = (1-\beta)E_n+\beta E_{n-1}$. Therefore, the calculation required by equation III.C.3.1 is straightforward once the density of Y is obtained. We have, with $f_{E_{n-1}}(x)$ the p.d.f. of E_{n-1} (i.e. e^{-x})

$$P([1-\beta]E_n+\beta E_{n-1} \le y) = \int_0^{y/\beta} P([1-\beta]E_n+\beta x \le y \mid E_{n-1} = x) f_{E_{n-1}}(x)dx$$

$$= \int_0^{y/\beta} P([1-\beta]E_n \le y-\beta x \mid E_{n-1} = x) f_{E_{n-1}}(x)dx$$

$$= \int_0^{y/\beta} P(E_n \le \frac{y-\beta x}{1-\beta} \mid E_{n-1} = x) f_{E_{n-1}}(x)dx$$

$$= \int_0^{y/\beta} (1 - e^{-\frac{y-\beta x}{1-\beta}}) e^{-x}dx$$

$$= \int_0^{y/\beta} e^{-x}dx - \int_0^{y/\beta} e^{-\frac{y}{1-\beta}} e^{-[x-\frac{\beta x}{1-\beta}]} dx$$

$$= 1 - e^{-y/\beta} - e^{-\frac{y}{1-\beta}} \int_0^{y/\beta} e^{-x[\frac{1-2\beta}{1-\beta}]} dx$$

$P([1-\beta]E_n+\beta E_{n-1} \leq y)$

$$= 1 - e^{-y/\beta} - e^{-\frac{y}{1-\beta}}[\tfrac{1-\beta}{1-2\beta}] \int_0^{y/\beta} [\tfrac{1-2\beta}{1-\beta}]e^{-x[\frac{1-2\beta}{1-\beta}]}dx$$

$$= 1 - e^{-y/\beta} - e^{-\frac{y}{1-\beta}}[\tfrac{1-\beta}{1-2\beta}](1-e^{-\frac{y}{\beta}[\frac{1-2\beta}{1-\beta}]})$$

$$= 1 - e^{-y/\beta} - [\tfrac{1-\beta}{1-2\beta}]e^{-\frac{y}{1-\beta}} + [\tfrac{1-\beta}{1-2\beta}]e^{-y/\beta}$$

$$P(Y \leq y) \quad = \quad -\frac{\beta}{1-2\beta}[1-e^{-y/\beta}] + \frac{1-\beta}{1-2\beta}(1-e^{-\frac{y}{1-\beta}})$$

Therefore, the density function of Y, $f_Y(y)$, is $-(\frac{\beta}{1-2\beta})(\frac{1}{\beta})e^{-y/\beta}+(\frac{1-\beta}{1-2\beta})(\frac{1}{1-\beta}\ e^{-y/(1-\beta)}$, for $\beta \neq \frac{1}{2}$. Gaver and Lewis [Ref. 2] gave the necessary and sufficient conditions for a mixed exponential of the form $\pi_1\lambda_1 e^{-\lambda_1 x}+\pi_2\lambda_2 e^{-\lambda_2 x}$, $\pi_1 \geq 1$, $\pi_1 + \pi_2 = 1$, and $\lambda_1 < \lambda_2$ to be a proper density function. The condition is that

$$\pi_1 \quad \leq \quad (1 - \frac{\lambda_1}{\lambda_2})^{-1} \qquad \text{(III.C.3.2)}$$

In this situation we address two separate cases. The first case is when $0 < \beta < \frac{1}{2}$. In this case $\frac{1-\beta}{1-2\beta} > 1$ and the requirement is

$$\frac{1-\beta}{1-2\beta} \;\leq\; [1 - \{(\tfrac{\lambda}{1-\beta}) \div (\tfrac{\lambda}{\beta})\}]^{-1} \;\leq\; [1 - \tfrac{\beta}{1-\beta}]^{-1}$$

$$\leq \frac{1-\beta}{1-2\beta}$$

And III.C.3.2 is satisfied. The second case is when $\frac{1}{2} < \beta < 1$. In this case $-\frac{\beta}{1-2\beta} > 1$ and the requirement is

$$-\frac{\beta}{1-2\beta} \leq [1 - \{(\tfrac{1}{\beta}) \div (\tfrac{1}{1-\beta})\}]^{-1}$$

$$\leq [1 - \tfrac{1-\beta}{\beta}]^{-1}$$

$$\leq -\frac{\beta}{1-2\beta}$$

And III.C.3.2 is satisfied. If $\beta = \frac{1}{2}$, the density of Y is a Gamma(2) density. Therefore, the p.d.f. of Y is a proper density. This result can be used to complete the calculation of $P(X_{n+1} > X_n)$. Recall that equation IIIC.3.1 stated

$$P(X_{n+1} > X_n) = P(E_{n+1} > (1-\beta)E_n + \beta E_{n-1})$$

$$= P(E_{n+1} > Y)$$

By conditioning on Y and using the p.d.f. for Y derived above this is

$$P(X_{n+1} > X_n) = \int_0^\infty P(E_{n+1} > y \mid Y = y) f_Y(y)\,dy$$

$$= \int_0^\infty e^{-y}[-(\tfrac{\beta}{1-2\beta})(\tfrac{1}{\beta})e^{-y/\beta} + (\tfrac{1-\beta}{1-2\beta})(\tfrac{1}{1-\beta})e^{-\frac{y}{1-\beta}}]\,dy$$

$$= -(\tfrac{1}{1-2\beta})(\tfrac{\beta}{\beta+1})\int_0^\infty (\tfrac{\beta+1}{\beta})e^{-y(\frac{\beta+1}{\beta})}\,dy$$

$$+ (\tfrac{1}{1-2\beta})(\tfrac{1-\beta}{2-\beta})\int_0^\infty (\tfrac{2-\beta}{1-\beta})e^{-(\frac{2-\beta}{1-\beta})}\,dy$$

$$P(X_{n+1} > X_n) = -\left(\frac{1}{1-2\beta}\right)\left(\frac{\beta}{\beta+1}\right) + \left(\frac{1}{1-2\beta}\right)\left(\frac{1-\beta}{2-\beta}\right)$$

$$P(X_{n+1} > X_n) = \frac{1}{(2-\beta)(1+\beta)} \qquad \text{(III.C.3.3)}$$

Table III.C.3.1 presents the results of the calculations for all of the sixteen combinations of parameter values for X_{n+1} and X_n.

When the $P(X_{n+1} > X_n)$ was calculated for various values of α and β, it was found that the values for this probability varied from 0.44 to 0.54. Table III.C.3.2 contains the results of these calculations for four hundred forty-one combinations of parameter values. Although the variation in probability is not large, it does represent an increase over the forward EMA(1) model. In particular, the forward EMA(1) model can not produce a probability greater than 0.50. Consequently, the NEMA(1) model not only has a greater range of possible probabilities, but also can produce probabilities greater than 0.50. The implications of this greater range is that the NEMA(1) model can address data sample paths that have a slight tendency for either runs of increasing or decreasing values, while the EMA(1) can only address sample paths that tend to produce runs of decreasing values.

Examples of scatter plots and sample paths for three sets of parameter values and positive correlations are given in Figures III.C.3.1-III.C.3.6. Because of the relatively low correlations possible and because of the limited range of values for $P(X_{n+1} > X_n)$, differences among the figures are not

TABLE III.C.3.1

K_{n+1}	K_n	I_{n+1}	I_n	Probability of Parameter Values Occurring	$P(X_{n+1} > X_n)$
1	1	β	β	$\delta^2\alpha^2$	$\frac{1}{(1+\beta)(2-\beta)}$
1	1	β	0	$\delta^2\alpha(1-\alpha)$	$\frac{1}{2-\beta}$
1	1	0	β	$\delta^2(1-\alpha)\alpha$	$\frac{1}{2(1+\beta)}$
1	1	0	0	$\delta^2(1-\alpha)^2$	$\frac{1}{2}$
1	$(1-\alpha)\beta$	β	β	$\delta(1-\delta)\alpha^2$	$\frac{1+\alpha+\alpha\beta}{(1+\alpha)(1+\beta)}$
1	$(1-\alpha)\beta$	β	0	$\delta(1-\delta)\alpha(1-\alpha)$	1
1	$(1-\alpha)\beta$	0	β	$\delta(1-\delta)(1-\alpha)\alpha$	$\frac{1}{(1+\beta)(1+\beta-\alpha\beta)}$
1	$(1-\alpha)\beta$	0	0	$\delta(1-\delta)(1-\alpha)^2$	$\frac{1}{1+\beta-\alpha\beta}$
$(1-\alpha)\beta$	1	β	β	$(1-\delta)\delta\alpha^2$	$\frac{(1-\alpha)^2\beta}{(1-\alpha\beta)(2-\alpha)}$
$(1-\alpha)\beta$	1	β	0	$(1-\delta)\delta\alpha(1-\alpha)$	$\frac{(1-\alpha)\beta}{1-\alpha\beta}$
$(1-\alpha)\beta$	1	0	β	$(1-\delta)\delta(1-\alpha)\alpha$	$\frac{\beta(1-\alpha)^2}{(1+\beta-\alpha\beta)(2-\alpha)}$
$(1-\alpha)\beta$	1	0	0	$(1-\delta)\delta(1-\alpha)^2$	$\frac{(1-\alpha)\beta}{1+\beta-\alpha\beta}$
$(1-\alpha)\beta$	$(1-\alpha)\beta$	β	β	$(1-\delta)^2\alpha^2$	$\frac{1}{(2-\alpha)(1+\alpha)}$
$(1-\alpha)\beta$	$(1-\alpha)\beta$	β	0	$(1-\delta)^2\alpha(1-\alpha)$	1
$(1-\alpha)\beta$	$(1-\alpha)\beta$	0	β	$(1-\delta)^2(1-\alpha)\alpha$	$\frac{1-\alpha}{2(2-\alpha)}$
$(1-\alpha)\beta$	$(1-\alpha)\beta$	0	0	$(1-\delta)^2(1-\alpha)^2$	$\frac{1}{2}$

TABLE III.C.3.2

ALPHA \ BETA	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.0	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
0.05	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.51	0.51	0.51	0.51	0.51
0.10	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.51	0.51	0.51	0.51	0.51	0.52	0.52	0.52
0.15	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.51	0.51	0.51	0.52	0.52	0.53	0.53	0.53
0.20	0.50	0.50	0.50	0.49	0.49	0.49	0.49	0.49	0.50	0.50	0.50	0.50	0.51	0.51	0.51	0.52	0.52	0.53	0.53	0.53	0.53
0.25	0.50	0.50	0.50	0.49	0.49	0.49	0.49	0.49	0.49	0.50	0.50	0.50	0.51	0.51	0.52	0.52	0.53	0.53	0.54	0.54	0.54
0.30	0.50	0.50	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.50	0.50	0.50	0.51	0.51	0.52	0.52	0.53	0.53	0.54	0.54	0.54
0.35	0.50	0.50	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.50	0.50	0.50	0.51	0.52	0.52	0.53	0.53	0.54	0.54	0.54
0.40	0.50	0.50	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.49	0.50	0.50	0.51	0.51	0.52	0.53	0.53	0.54	0.54	0.54
0.45	0.50	0.49	0.49	0.49	0.49	0.48	0.48	0.48	0.49	0.49	0.49	0.49	0.50	0.51	0.51	0.52	0.52	0.53	0.53	0.54	0.53
0.50	0.50	0.49	0.49	0.49	0.48	0.48	0.48	0.48	0.48	0.48	0.49	0.49	0.50	0.50	0.51	0.51	0.52	0.52	0.53	0.53	0.53
0.55	0.50	0.49	0.49	0.48	0.48	0.48	0.48	0.48	0.48	0.48	0.48	0.49	0.49	0.50	0.50	0.51	0.51	0.52	0.52	0.52	0.52
0.60	0.50	0.49	0.49	0.48	0.48	0.48	0.47	0.47	0.47	0.48	0.48	0.48	0.48	0.49	0.49	0.50	0.51	0.51	0.52	0.52	0.51
0.65	0.50	0.49	0.49	0.48	0.48	0.47	0.47	0.47	0.47	0.47	0.47	0.47	0.48	0.48	0.49	0.49	0.50	0.50	0.51	0.51	0.51
0.70	0.50	0.49	0.48	0.48	0.47	0.47	0.47	0.47	0.46	0.46	0.47	0.47	0.47	0.47	0.48	0.48	0.49	0.49	0.50	0.50	0.50
0.75	0.50	0.49	0.48	0.48	0.47	0.47	0.46	0.46	0.46	0.46	0.46	0.46	0.46	0.47	0.47	0.48	0.48	0.49	0.49	0.49	0.49
0.80	0.50	0.49	0.48	0.48	0.47	0.46	0.46	0.46	0.46	0.45	0.45	0.45	0.46	0.46	0.46	0.47	0.47	0.48	0.48	0.49	0.49
0.85	0.50	0.49	0.48	0.47	0.47	0.46	0.46	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.46	0.46	0.46	0.47	0.48	0.48	0.49
0.90	0.50	0.49	0.48	0.47	0.47	0.46	0.45	0.45	0.45	0.45	0.44	0.44	0.45	0.45	0.45	0.45	0.46	0.47	0.47	0.48	0.49
0.95	0.50	0.49	0.48	0.47	0.46	0.46	0.45	0.45	0.45	0.44	0.44	0.44	0.44	0.45	0.45	0.45	0.46	0.46	0.47	0.48	0.49
1.00	0.50	0.49	0.48	0.47	0.46	0.46	0.45	0.45	0.45	0.44	0.44	0.44	0.45	0.45	0.45	0.46	0.46	0.47	0.48	0.49	0.0

sharply delineated. However, differences can be detected, particularly in the sample paths. In Figure III.C.3.1 with an α value of 0.95 and β value of 0.50 has a $P(X_{n+1} > X_n) =$ 0.44, the lowest value for this probability. Since this produces a slight tendency for runs of decreasing value, the number of extreme values (i.e. greater than 3.0) is two. In Figures III.C.3.2 and IIIC.3.3 the $P(X_{n+1} > X_n)$ is 0.50 and 0.54, respectively, with a corresponding increase in the number of large values. This trend is more difficult to detect in the corresponding scatter plots. Figures III.C.3.7-III.C.3.12 provide sample paths and scatter plots for the same α and β values as previously displayed, but with antithetic innovative sequences (see III.B) and consequent negative correlations. Although the negative correlation is evident, trends in these figures are difficult to detect. The extremes of sample path variability produced by the NEAR(1) process [Ref. 8] are not reproducible with the NEMA(1) process. This may be attributable to the restricted range of possible correlations.

4. Laplace Transform of Sums

One aspect of the EMA(1) model is its analytical tractability. This is evidenced by the ability to derive the Laplace transform of sums by a recursive relationship given by Lawrance and Lewis [Ref. 5]. This tractability carries over to the NEMA(1) process. The Laplace transform is useful in obtaining quantities which are of use in analyzing point processes, namely the intensity function and the spectrum of

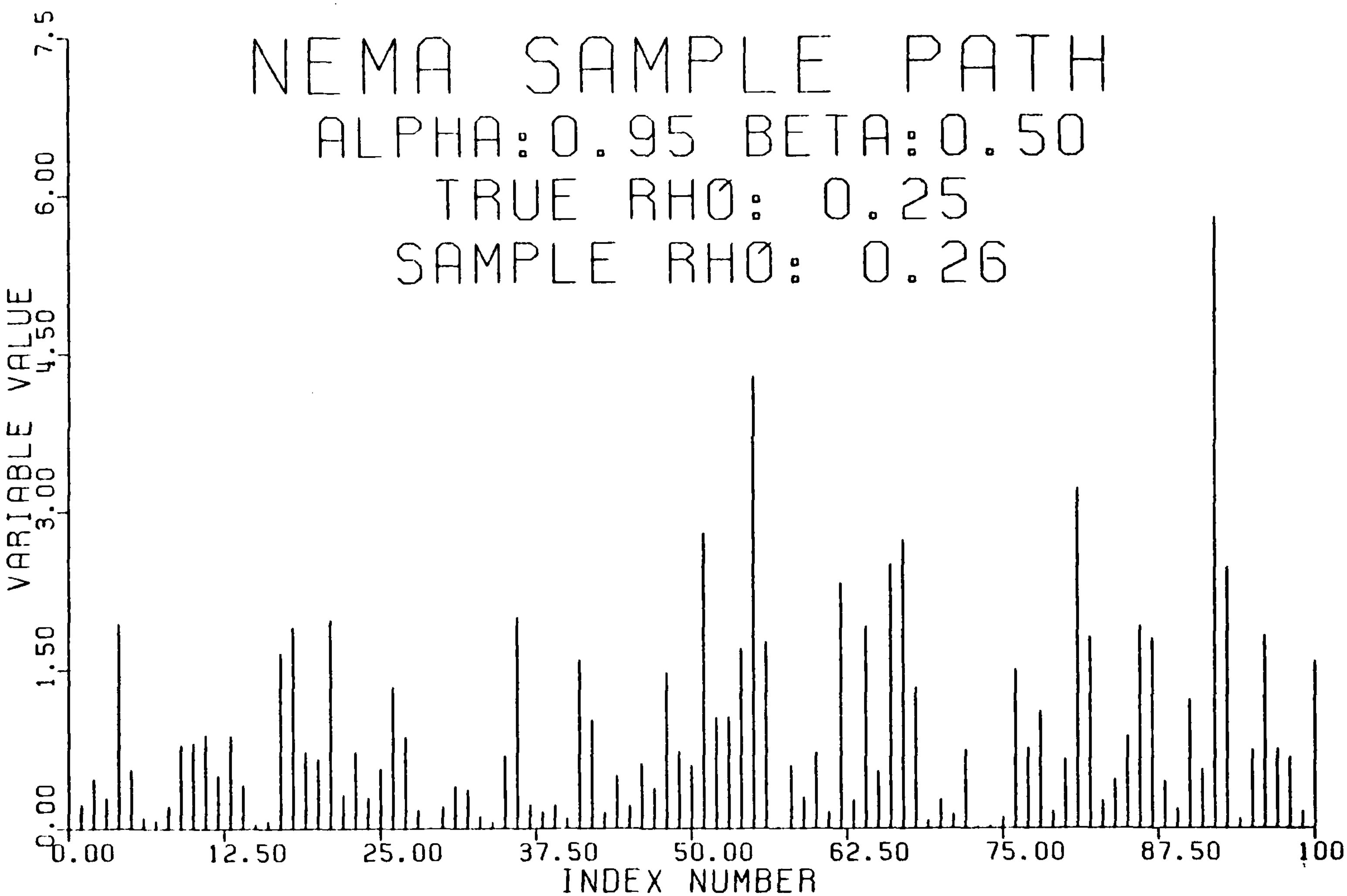


FIGURE III.C.3.1

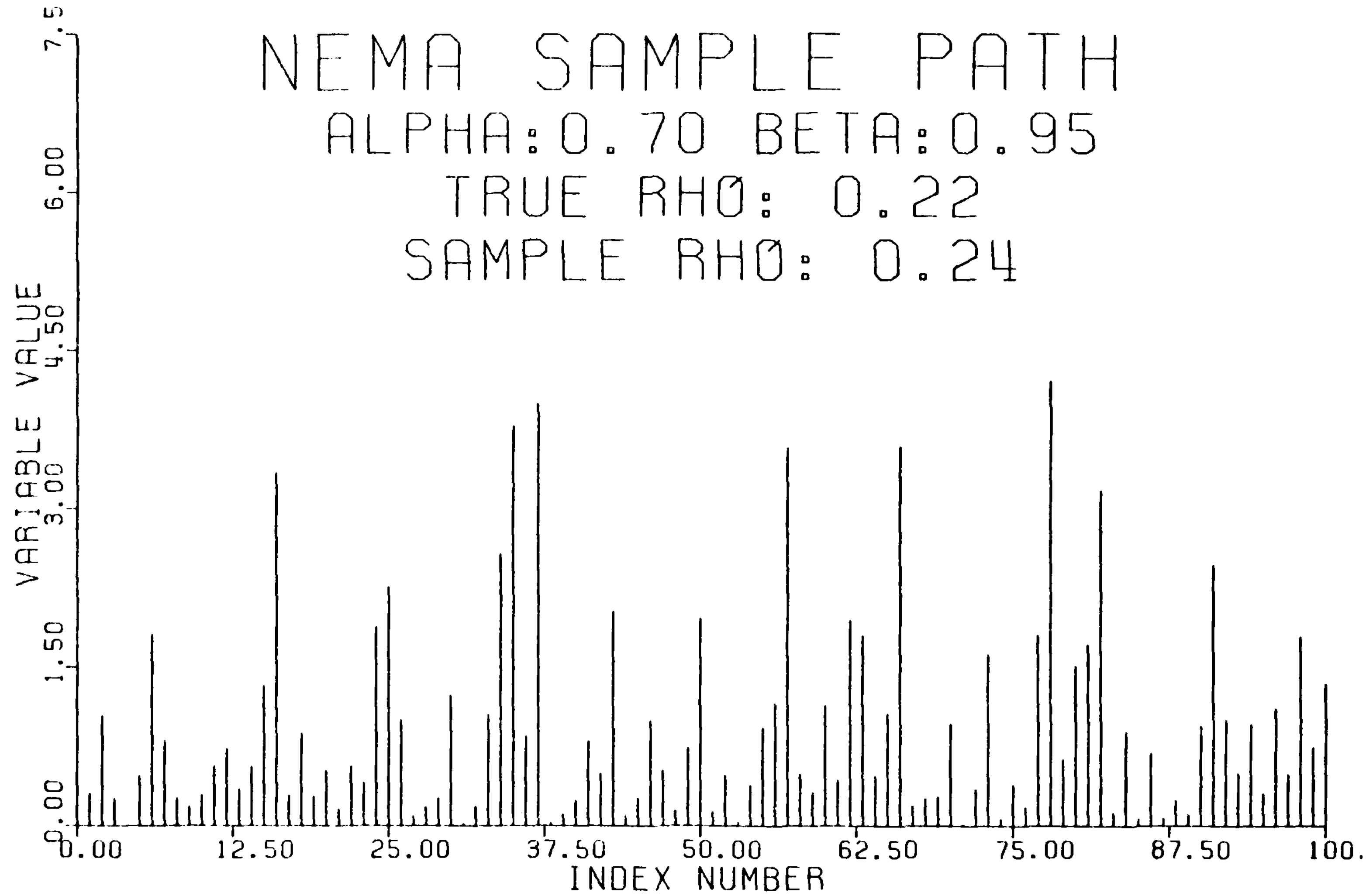


FIGURE III.C.3.2

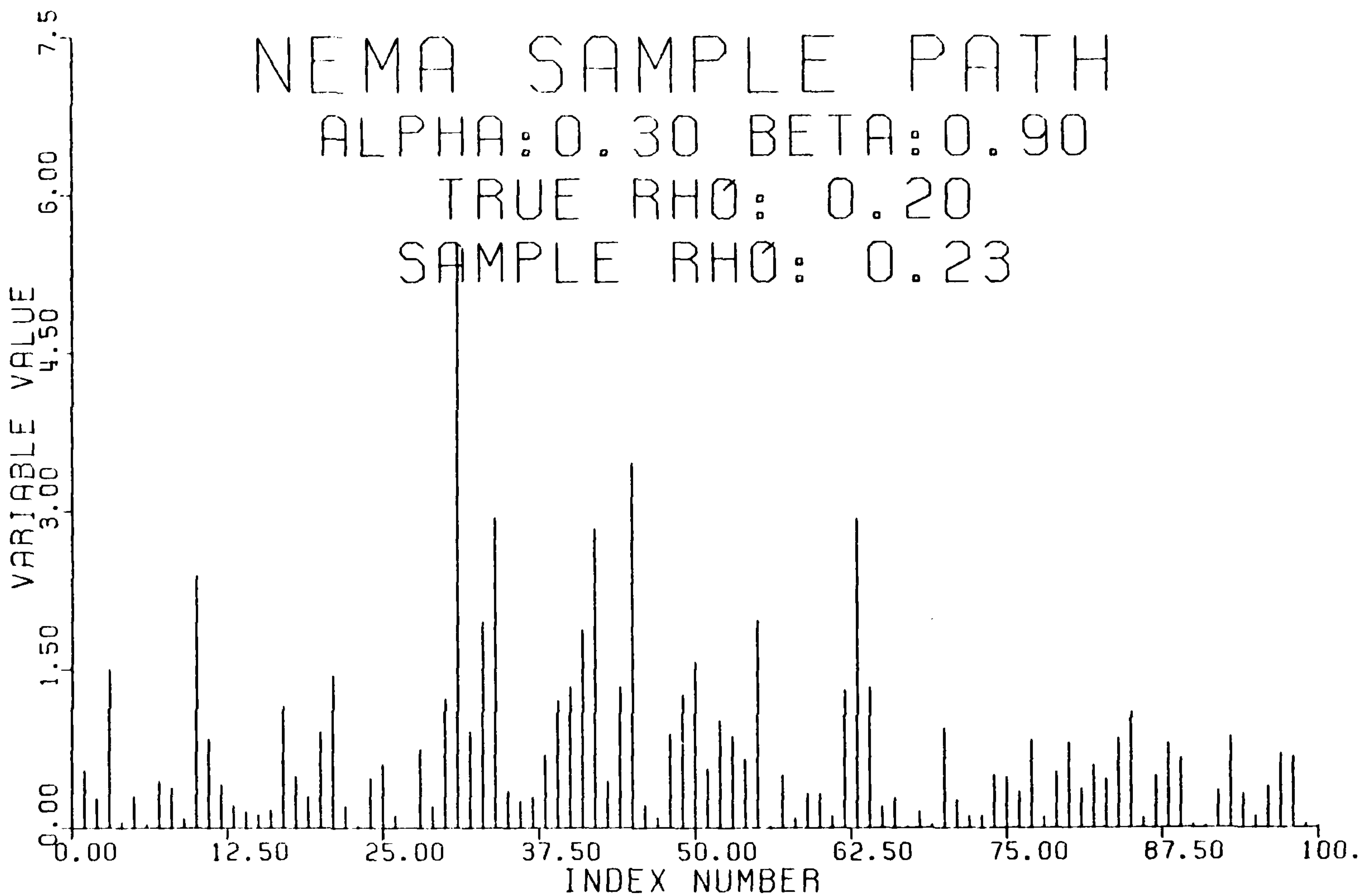


FIGURE III.C.3.3

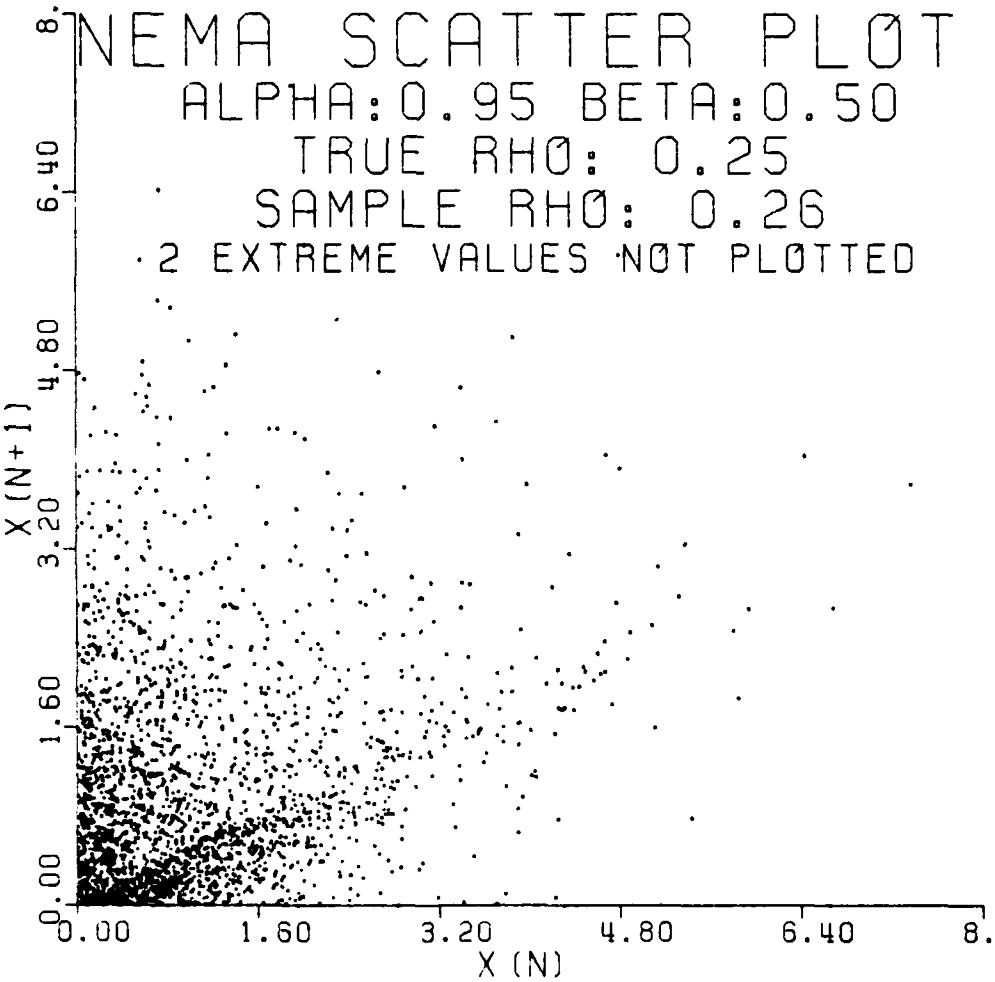


FIGURE III.C.3.4

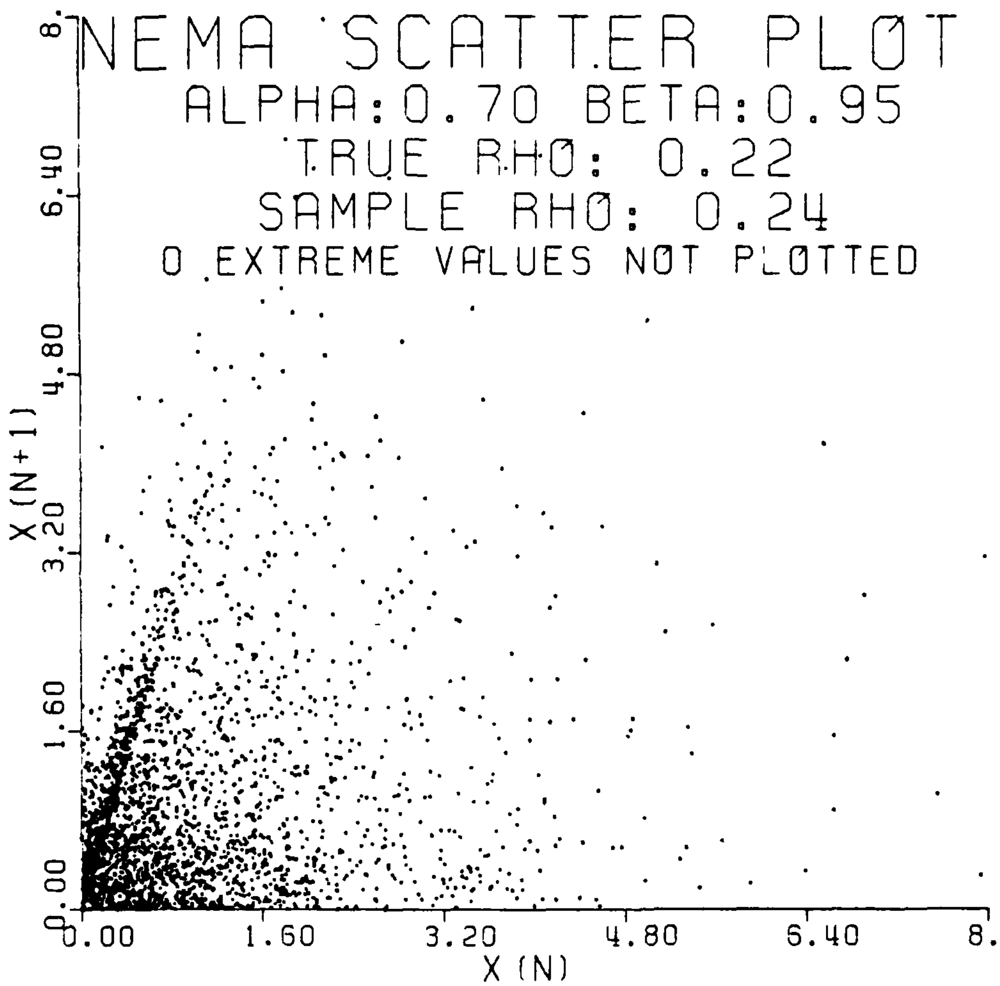


FIGURE III.C.3.5

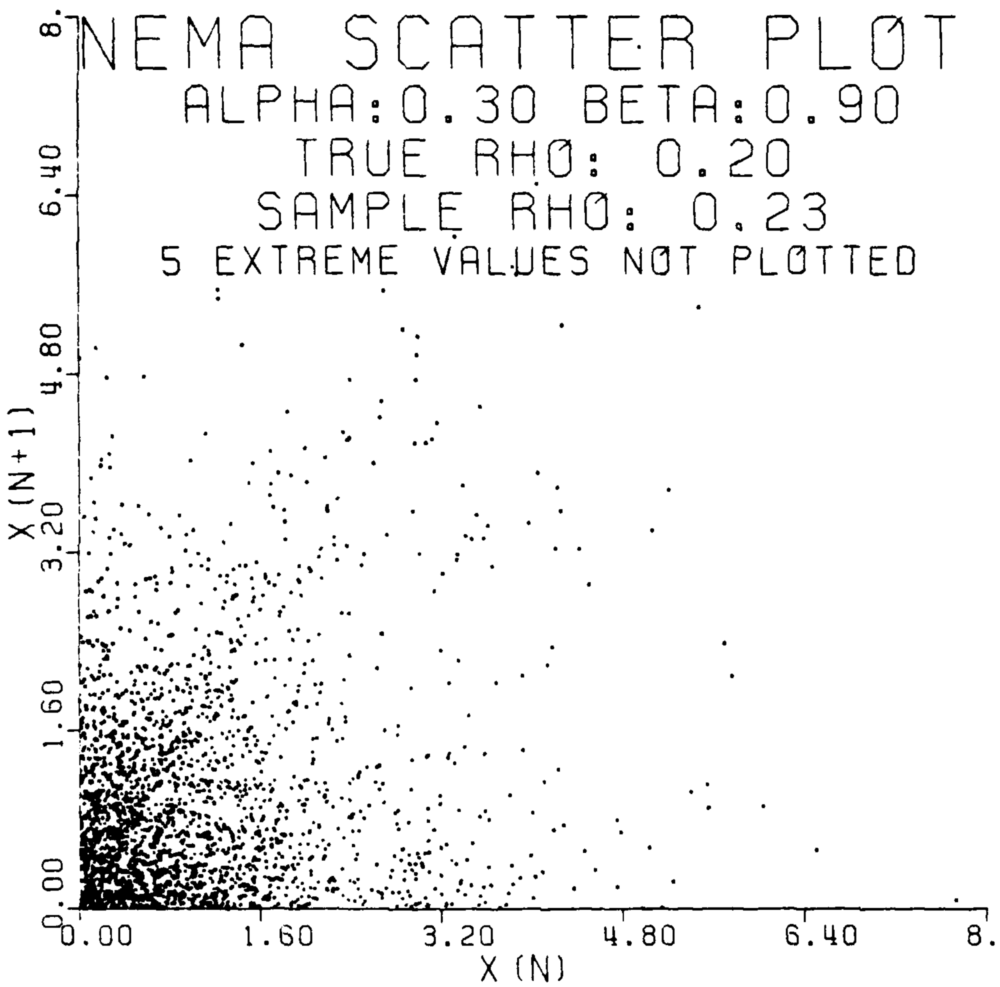


FIGURE III.C.3.6

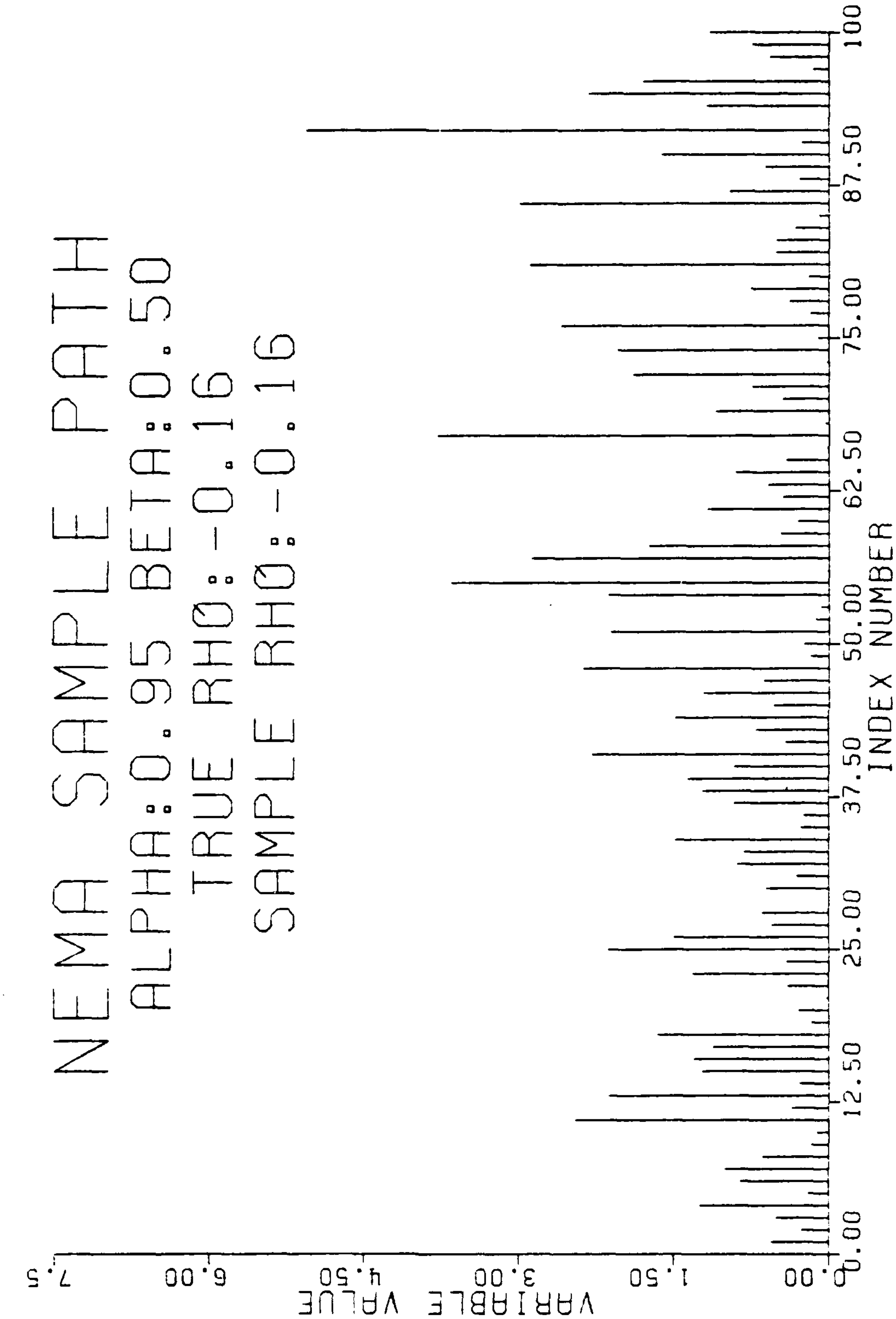


FIGURE III.C.3.7

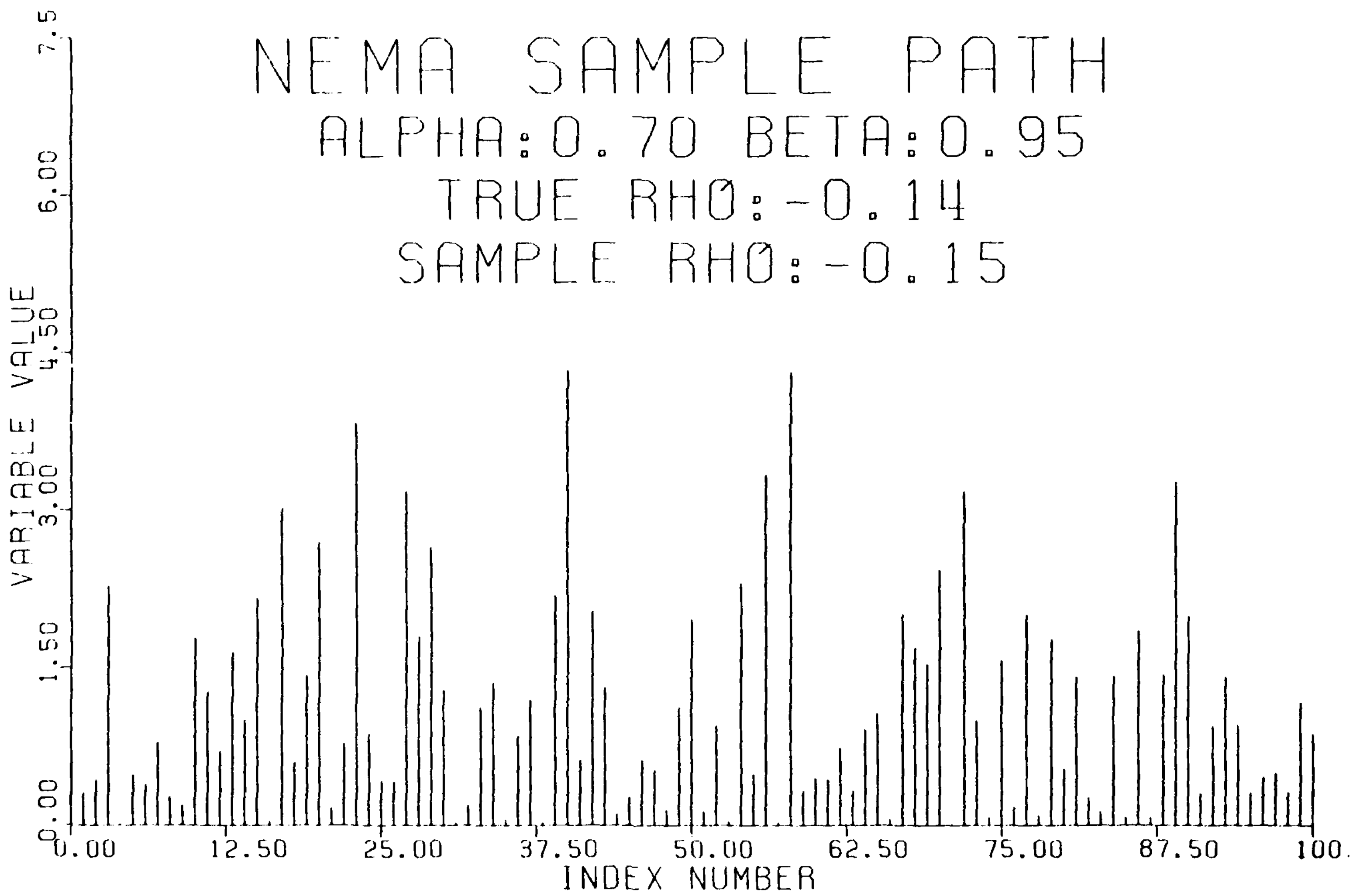


FIGURE III.C.3.8

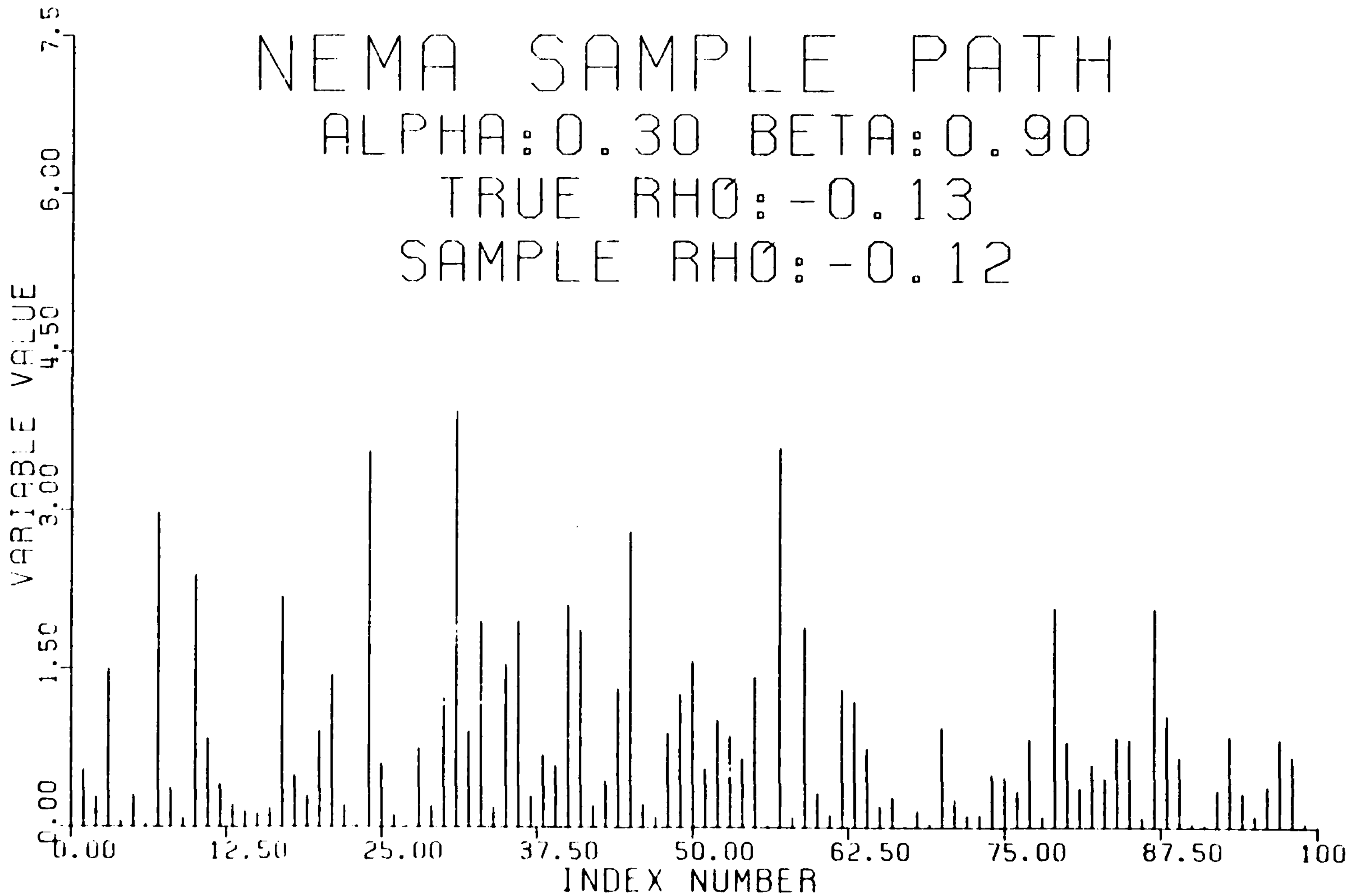


FIGURE III.C.3.9

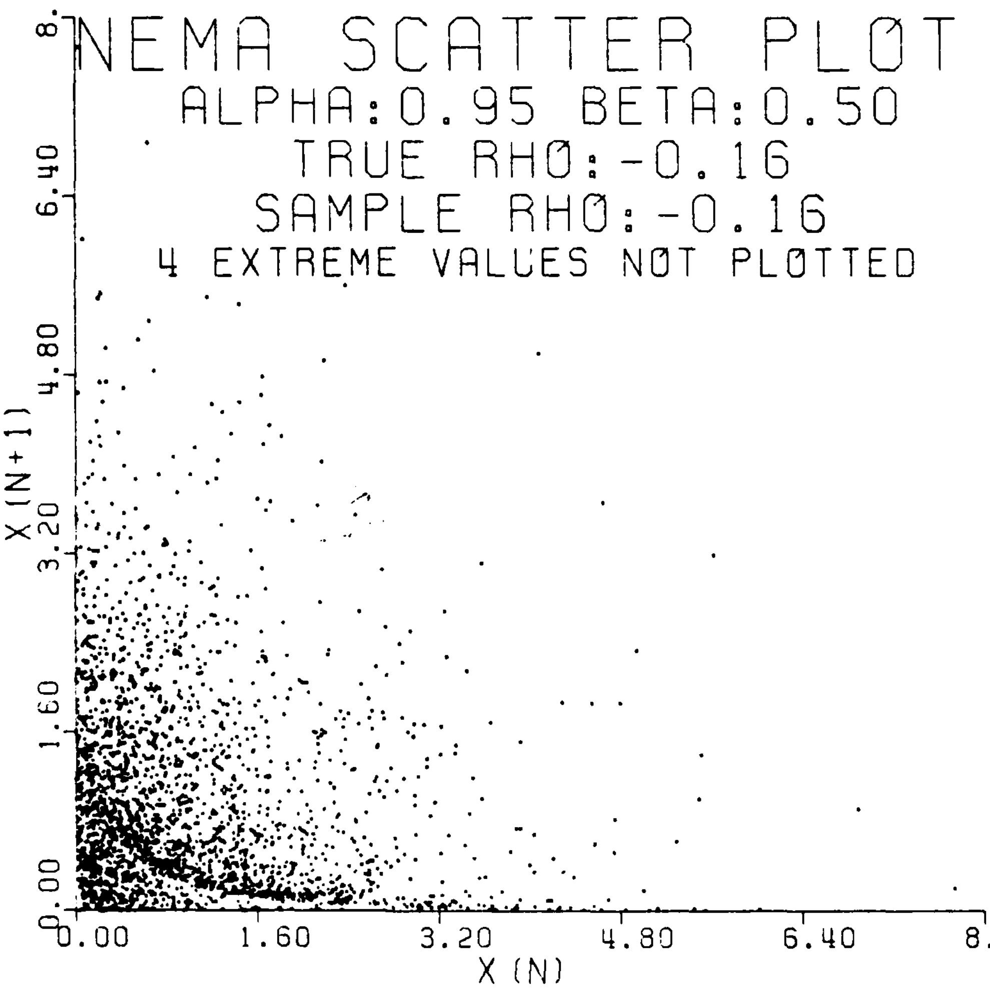


FIGURE III.C.3.10

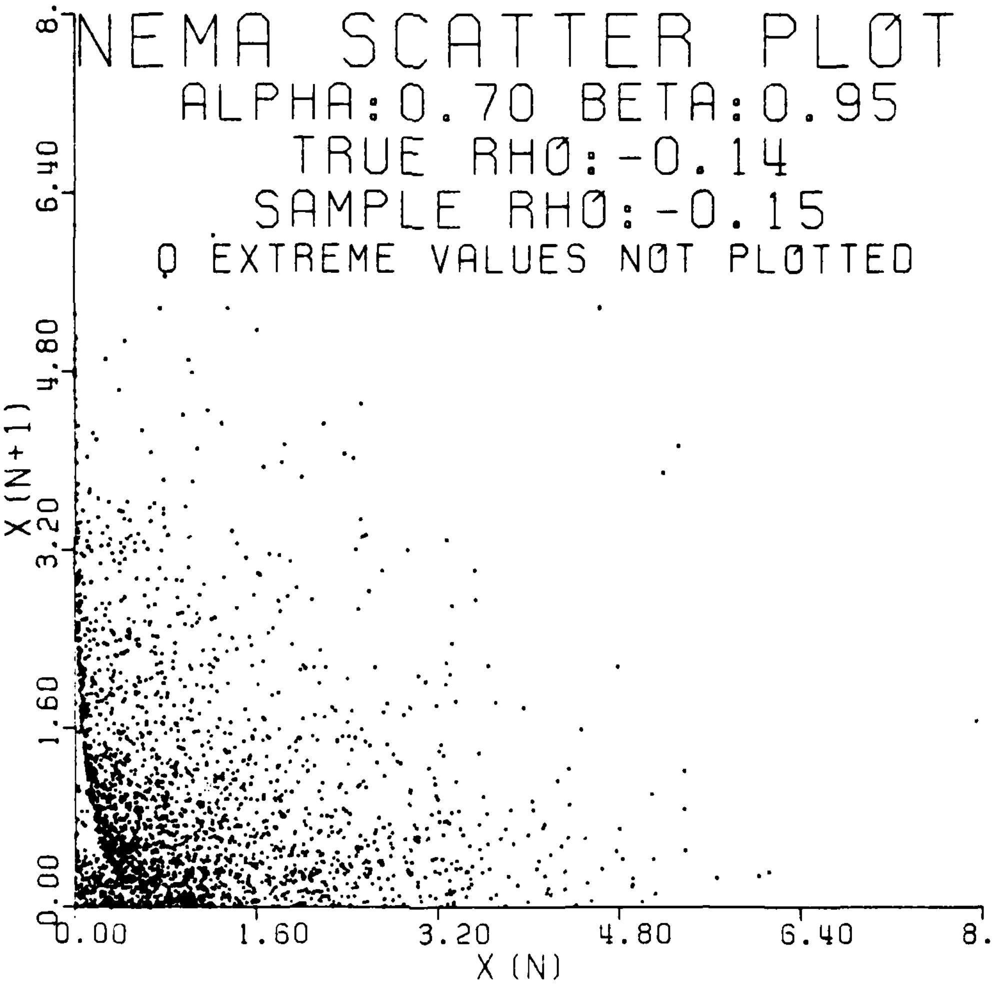


FIGURE III.C.3.11

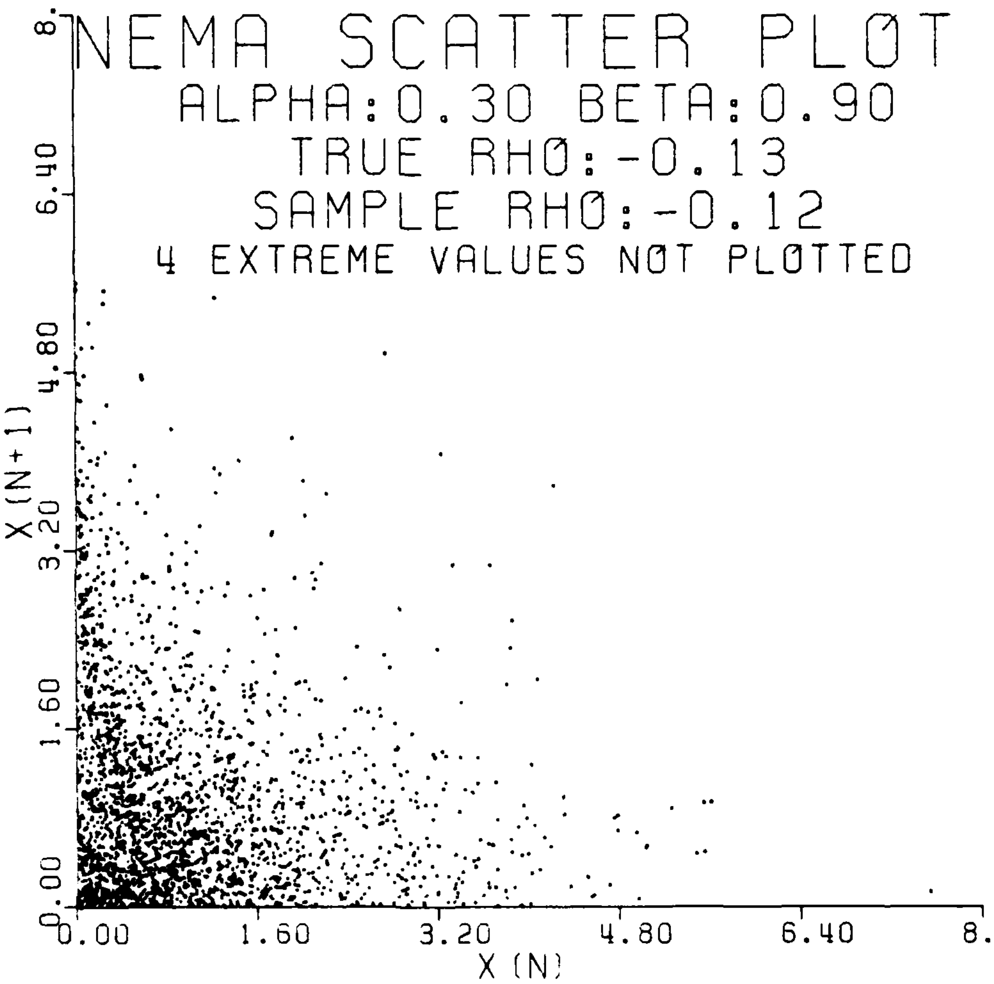


FIGURE II.C.3.12

counts. These quantities are derived for the NEMA(1) process in subsequent sections from the results obtained here.

With X_n defined as in III.C.1.5, let $\frac{1-\beta}{1-(1-\alpha)\beta} = \delta$ and $(1-\alpha)\beta = \gamma$. Further, let $\sum_{i=1}^{r} X_i = T_r$ and $\phi_{T_r}(s) = E(e^{-sT_r})$. Then we have

$$T_r = X_1 + X_2 + \dots + X_r \tag{III.C.4.1}$$

$$= K_1E_1 + I_1E_0 + K_2E_2 + I_2E_1 + \dots + K_rE_r + I_rE_{r-1}$$

$$= K_rE_r + (I_r + K_{r-1})E_{r-1} + \dots + (I_2 + K_1)E_1 + I_1E_0$$

Then letting $L_j = I_{j+1} + K_j$, $j = 1,2,\dots,r-1$ and using the mutual independence of the iid sequences $\{K_n\}$, $\{I_n\}$, $\{E_n\}$

$$\phi_{T_r}(s) = E(e^{-sT_r}) \tag{III.C.4.2}$$

$$= E(e^{-s[K_rE_r+L_{r-1}E_{r-1}+\dots+L_1E_1+I_1E_0]})$$

$$= E(e^{-sK_rE_r})E(e^{-sL_{r-1}E_{r-1}})\dots E(e^{-sL_1E_1})E(e^{-sI_1E_0})$$

Now let $\Psi_K(s) = E(e^{-sK_jE_j})$, $\Psi_I(s) = E(e^{-sI_jE_j})$, $\Psi_L(s) = E(e^{-sL_jE_j})$. Then

$$\phi_{T_r}(s) = \Psi_K(s)\Psi_I(s)[\Psi_L(s)]^{r-1} \tag{III.C.4.3}$$

To evaluate these quantities note that

$$K = \begin{cases} 1 & \text{with probability } \delta, \\ \gamma & \text{with probability } 1-\delta. \end{cases}$$

Then

$$\Psi_K(s) = E(e^{-sK_jE_j})$$

$$= \delta E(e^{-sE_j}) + (1-\delta)E(e^{-s\gamma E_j})$$

$$\Psi_K(s) = \delta\phi_E(s) + (1-\delta)\phi_E(\gamma s), \qquad \text{(III.C.4.4)}$$

where $\phi_E(s) = \frac{1}{1+s}$. So

$$\Psi_K(s) = \frac{\delta}{1+s} + \frac{(1-\delta)}{1+\gamma s} \qquad \text{(III.C.4.5)}$$

Also

$$I = \begin{cases} \beta & \text{with probability } \alpha, \\ 0 & \text{with probability } 1-\alpha, \end{cases}$$

so that

$$\Psi_I(s) = E(e^{-sI_jE_j}) = \alpha E(e^{-s\beta E_j}) + (1-\alpha) \qquad \text{(III.C.4.6}$$

$$\Psi_I(s) = \frac{\alpha}{1+\beta s} + (1-\alpha). \qquad \text{(III.C.4.7)}$$

To evaluate $\Psi_L(s)$ note that

$$L = \begin{cases} \beta+1 & \text{with probability } \alpha\delta, \\ \beta+\gamma & \text{with probability } \alpha(1-\delta), \\ 1 & \text{with probability } (1-\alpha)\delta, \\ \gamma & \text{with probability } (1-\alpha)(1-\delta). \end{cases}$$

Therefore

$$\begin{aligned} \Psi_L(s) &= E(e^{-sL_jE_j}) \\ &= \alpha\delta E(e^{-s[\beta+1]E_j}) + \alpha(1-\delta)E(e^{-s[\beta+\gamma]E_j}) \\ &\quad + (1-\alpha)\delta E(e^{-sE_j}) + (1-\alpha)(1-\delta)E(e^{-s\gamma E_j}) \end{aligned}$$

$$\begin{aligned} \Psi_L(s) &= \alpha\delta\phi_E([\beta+1]s) + \alpha(1-\delta)\phi_E([\beta+\gamma]s) + (1-\alpha)\delta\phi_E(s) \\ &\quad + (1-\alpha)(1-\delta)\phi_E(\gamma s). \end{aligned} \tag{III.C.4.8}$$

Using the results of III.C.4.3, III.C.4.4, III.C.4.7, and III.C.4.8

$$\begin{aligned} \phi_{T_r}(s) &= [\delta\phi_E(s)+(1-\delta)\phi_E(\gamma s)]\times[\alpha\phi_E(\beta s)+(1-\alpha)] \\ &\quad \times[\alpha\delta\phi_E([\beta+1)s)+\alpha(1-\delta)\phi_E([\beta+\gamma]s)+(1-\alpha)\delta\phi_E(s) \\ &\quad + (1-\alpha)(1-\delta)\phi_E(\gamma s)]^{r-1} \end{aligned} \tag{III.C.4.9}$$

This extends the result (3.7) in Lawrance and Lewis [Ref. 5] to the NEMA(1) process.

5. Laplace Transform of the Distribution of Counts

The Laplace transform of the sum is useful in deriving the distribution of the synchronous counting process of the number of events that occur in $(0,t]$ when the origin is established at the occurrence of an arbitrary event. The number of events in $(0,t]$ is related to the distribution of a sum by the relationship

$$N_t^f < r \quad \text{iff} \quad T_r > t, \quad r = 1,2,\ldots \qquad \text{(III.C.5.1)}$$

where N_t^f is the number of events in $(0,t]$ and T_r is the sum of the first interevent times. Thus

$$P(N_t^f = r) = F_r(t) - F_{r+1}(t)$$

where $F_r(t)$ is the distribution function of T_r. The probability generating function of N_t^f can then be written as

$$\begin{aligned}
\Psi_f(z;t) &= E(z^{N_t^f}) \\
&= \sum_{r=0}^{\infty} z^r P(N_t^f = r) \\
&= \sum_{r=0}^{\infty} z^r [F_r(t) - F_{r+1}(t)] \\
&= 1 + (z-1) \sum_{r=1}^{\infty} z^{r-1} F_r(t).
\end{aligned}$$

Let $\Psi_f^*(z;s)$ be the Laplace transform of $\Psi_f(z;t)$, and $f_r^*(t)$ the Laplace transform of $f_r(t)$, the p.d.f. of T_r. Then

$$\Psi_f^*(z;s) = \frac{1}{s} - \frac{(1-z)}{s} \sum_{r=1}^{\infty} z^{r-1} f_r^*(t) \qquad \text{(III.C.5.2)}$$

Using the Laplace transform of the sum from III.C.4.9

$$\begin{aligned}\Psi_f^*(z;s) = \frac{1}{s} - \frac{(1-z)}{s} \sum_{r=1}^{\infty} z^{r-1} &[\delta\phi_E(s) + (1-\delta)\phi_E(\gamma s)] \\ &\times [\alpha\phi_E(\beta s) + (1-\alpha)] \times [\alpha\delta\phi_E([\beta+1]s) + \alpha(1-\delta)\phi_E([\beta+\gamma]s) \\ &+ (1-\alpha)\delta\phi_E(s) + (1-\alpha)(1-\delta)\phi_E(\gamma s)]^{r-1}\end{aligned}$$

$$\begin{aligned}\Psi_f^*(z;s) = \frac{1}{s} - \frac{(1-z)}{s} &[\delta\phi_E(s) + (1-\delta)\phi_E(\gamma s)] \times [\alpha\phi_E(\beta s) \\ &+ (1-\alpha)] \times \left\{ \frac{1}{1 - z[\alpha\delta\phi_E([\beta+1]s + \alpha(1-\delta)\phi_E([\beta+\gamma]s) + (1-\alpha)\delta\phi_E(s) + (1-\alpha)(1-\delta)\phi_E(\gamma s)]} \right\}\end{aligned} \qquad \text{(III.C.5.3)}$$

where $\phi_E(s) = \frac{1}{1+s}$.

If $m_f(t)$ is the intensity function of the point process, then $m_f^*(t)$, its Laplace transform, can be obtained by differentiating III.C.5.3 with respect to z, evaluating the derivative at z = 1, and then differentiating with respect to s. These steps, when taken, produce a series of tedious calculations which produce no analytical insights. The result of these steps is

$$m_f^* = \frac{1+(1+4\beta-2\alpha\beta)s+(3\beta+5\beta^2-2\alpha\beta-5\alpha\beta^2+\alpha^2\beta^2)s^2 +(2\beta^2+2\beta^3-3\alpha\beta^2-3\alpha\beta^3+\alpha^2\beta^2+\alpha^2\beta^3)s^3}{s+(1+4\beta-3\alpha\beta+\alpha^2\beta^2)s^2+(3\beta+5\beta^2-2\alpha\beta-7\alpha\beta^2+3\alpha^2\beta^2)s^3 +(2\beta^2+2\beta^3-3\alpha\beta^2-3\alpha\beta^3+\alpha^2\beta^2+\alpha^2\beta^3)s^4} \qquad \text{(III.C.5.4)}$$

This result can be verified in a number of ways. First, when $\alpha = 1$, the process is the EMA(1) process and, hence, the formula must reduce to (4.2) given in Lawrance and Lewis [Ref. 5] with $\lambda = 1$. Second, when $\alpha = 0$, the NEMA(1) process reduces to a Poisson process and the formula under this condition must reduce to the Laplace transform of the constant intensity function of a Poisson process with rate 1, $\frac{1}{s}$. Third, with $\beta = 0$ the NEMA(1) process is again a Poisson process. Finally, using one of the Tauberian Theorems, $\lim_{t\to\infty} m_f(t) = \lim_{s\to 0} s m_f^*(s) = 1$. We take these cases in turn. First, when $\alpha = 1$

$$m_f^*(s) = \frac{1+(1+4\beta-2\alpha\beta)s+(3\beta+5\beta^2-2\alpha\beta-5\alpha\beta^2+\alpha^2\beta^2)s^2 +(2\beta^2+2\beta^3-3\alpha\beta^2-3\alpha\beta^3+\alpha^2\beta^2+\alpha^2\beta^3)s^3}{s+(1+4\beta-3\alpha\beta+\alpha^2\beta^2)s^2+(3\beta+5\beta^2-2\alpha\beta-7\alpha\beta^2+3\alpha^2\beta^2)s^3 +(2\beta^2+2\beta^3-3\alpha\beta^2-3\alpha\beta^3+\alpha^2\beta^2+\alpha^2\beta^3)s^4}$$

reduces to

$$m_f^*(s) = \frac{1+(1+2\beta)s+(\beta+\beta^2)s^2}{s+(1+\beta+\beta^2)s^2+(\beta+\beta^2)s^3}$$

$$m_f^*(s) = \frac{(1+\beta s)(1+[1+\beta]s)}{s([1+s][1+(\beta+\beta^2)s])}$$

$$= \frac{(1+\beta s)(1+[1+\beta]s)}{\beta(1+\beta)s(1+s)(\frac{1}{\beta(1+\beta)}+s)},$$

which is the result of Lawrance and Lewis [Ref. 5] with $\lambda = 1$.

In the second case with $\alpha = 0$

$$m_f^*(s) = \frac{1+(1+4\beta)s+(3\beta+5\beta^2)s^2+(2\beta^2+2\beta^3)s^3}{s+(1+4\beta)s^2+(3\beta+5\beta^2)s^3+(2\beta^2+2\beta^3)s^4}$$

$$= \frac{1}{s},$$

the Laplace transform of a Poisson process with rate of 1.

In the third case $\beta = 0$, so

$$m_f^*(s) = \frac{1+s}{s+s^2} = \frac{1}{s},$$

again the Laplace transform of a Poisson process with a rate of 1.

In the final case apply the Tauberian Theorem

$$\lim_{s\to 0} sm_f^*(s) = \frac{1+(1+4\beta-2\alpha\beta)s+(3\beta+5\beta^2-2\alpha\beta-5\alpha\beta^2+\alpha^2\beta^2)s^2+(2\beta^2+2\beta^3-3\alpha\beta^2+\alpha^2\beta^2+\alpha^2\beta^3)s^3}{1+(1+4\beta-3\alpha\beta+\alpha^2\beta^2)s+(3\beta+5\beta^2-2\alpha\beta-7\alpha\beta^2+3\alpha^2\beta^2)s^2+(2\beta^2+2\beta^3-3\alpha\beta^2+\alpha^2\beta^2+\alpha^2\beta^3)s^3}$$

$$= \frac{1}{1},$$

as required.

6. The Spectrum of Counts

For the statistical analysis of series of events the most useful quantity associated with a process is the (Bartlett) spectrum of counts. The spectrum of counts, $g_+(\omega)$, is the Fourier transform of the covariance density of $N_f(t)$. It is related to the Laplace transform of the intensity function, $m_f^*(s)$, by the relationship derived by Cox and Lewis [Ref. 29]

$$g_+(\omega) \quad = \quad \frac{\lambda}{\pi}(1 + m_f^*[i\omega] + m_f^*[-i\omega]).$$

We now derive this for the NEMA(1) process using III.C.5.4. In that expression for $m_f^*(s)$, let

$$a_1 \quad = \quad 1 + 4\beta - 2\alpha\beta, \qquad \text{(III.C.6.1)}$$

$$a_2 \quad = \quad 3\beta + 5\beta^2 - 2\alpha\beta - 5\alpha\beta^2 + \alpha^2\beta^2, \qquad \text{(III.C.6.2)}$$

$$a_3 \quad = \quad 2\beta^2 + 2\beta^3 - 3\alpha\beta^2 - 3\alpha\beta^3 + \alpha^2\beta^2 + \alpha^2\beta^3, \qquad \text{(III.C.6.3)}$$

$$b_1 \quad = \quad 1 + 4\beta - 3\alpha\beta + \alpha^2\beta^2, \qquad \text{(III.C.6.4)}$$

$$b_2 \quad = \quad 3\beta + 5\beta^2 - 2\alpha\beta - 7\alpha\beta^2 + 3\alpha^2\beta^2, \qquad \text{(III.C.6.5)}$$

$$b_3 \quad = \quad 2\beta^2 + 2\beta^3 - 3\alpha\beta^2 - 3\alpha\beta^3 + \alpha^2\beta^2 + \alpha^2\beta^3. \qquad \text{(III.C.6.6)}$$

Then

$$m_f^*(s) = \frac{1+a_1s+a_2s^2+a_3s^3}{s+b_1s^2+b_2s^3+b_3s^4}$$

Recall that $\lambda = 1$ so

$$g_+(\omega) = \frac{1}{\pi}\left[1+\frac{1+a_1(i\omega)+a_2(i\omega)^2+a_3(i\omega)^3}{i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4}\right.$$

$$\left.+\frac{1+a_1(-i\omega)+a_2(-i\omega)^2+a_3(-i\omega)^3}{-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4}\right]$$

$$g_+(\omega) = \frac{1}{\pi}\left\{\frac{[i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]\times[-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]}{[i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]\times[-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]}\right. \tag{III.C.6.7}$$

$$+\frac{[1+a_1(i\omega)+a_2(i\omega)^2+a_3(i\omega)^3]\times[-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]}{[i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]\times[-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]}$$

$$\left.+\frac{[1+a_1(-i\omega)+a_2(-i\omega)^2+a_3(-i\omega)^3]\times[i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]}{[i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]\times[-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]}\right\}$$

Consider the first term with numerator and denominator the same. Let the denominator = D.

$$D = [i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4][-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]$$

$$= \omega^2-ib_1\omega^3-b_2\omega^4+ib_3\omega^5+ib_1\omega^3+b_1^2\omega^4-ib_1b_2\omega^5-b_1b_3\omega^6-b_2\omega^4$$

$$+ib_1b_2\omega^5+b_2^2\omega^6-ib_2b_3\omega^7-ib_3\omega^5-b_1b_3\omega^6+ib_2b_3\omega^7+b_3^2\omega^8$$

$$= i(-b_1\omega^3+b_3\omega^5+b_1\omega^3-b_1b_2\omega^5+b_1b_2\omega^5-b_2b_3\omega^7+b_2b_3\omega^7-b_3\omega^5)$$

$$+(\omega^2-b_2\omega^4+b_1^2\omega^4-b_1b_3\omega^6-b_2\omega^4+b_2^2\omega^6-b_1b_3\omega^6+b_3^2\omega^8)$$

$$= \omega^2[1+[b_1^2-2b_2]\omega^2+[b_2^2-2b_1b_3]\omega^4+b_3^2\omega^6) \qquad \text{(III.C.6.8)}$$

Let

$$X = 1+(b_1^2-2b_2)\omega^2+(b_2^2-2b_1b_3)\omega^4+b_3^2\omega^6,$$

where b_1, b_2, and b_3 are defined by III.C.6.4, III.C.6.5, III.C.6.6, respectively. Then

$$D = \omega^2 X \qquad \text{(III.C.6.9)}$$

Consider the numerator of the second term in III.C.6.7 and call it N2. Then

$$N2 = [1+a_1(i\omega)+a_2(i\omega)^2+a_3(i\omega)^3][-i\omega+b_1(-i\omega)^2+b_2(-i\omega)^3+b_3(-i\omega)^4]$$

$$= -i\omega-b_1\omega^2+ib_2\omega^3+b_3\omega^4+a_1\omega^2-ia_1b_1\omega^3-a_1b_2\omega^4+ia_1b_3\omega^5+ia_2\omega^3$$

$$+a_2b_1\omega^4-ia_2b_2\omega^5-a_2b_3\omega^6-a_3\omega^4+ia_3b_1\omega^5+a_3b_2\omega^6-ia_3b_3\omega^7$$

$$N2 = i(-\omega+[a_2-a_1b_1+b_2]\omega^3+(a_1b_3-a_2b_2+a_3b_1)\omega^5-a_3b_3\omega^7) \quad \text{(III.C.6.10)}$$

$$+(a_1-b_1)\omega^2+(a_2b_1-a_1b_2-a_3+b_3)\omega^4+(a_3b_2-a_2b_3)\omega^6,$$

where a_1, a_2, a_3, b_1, b_2, and b_3 are defined in III.C.6.1 through III.C.6.6 respectively. Consider the numerator of the third term in III.C.6.7 and call it N3. Then

$$N3 = [1+a_1(-i\omega)+a_2(-i\omega)^2+a_3(-i\omega)^3][i\omega+b_1(i\omega)^2+b_2(i\omega)^3+b_3(i\omega)^4]$$

$$= i\omega-b_1\omega^2-ib_2\omega^3+b_3\omega^4+a_1\omega^2+ia_1b_1\omega^3-a_1b_2\omega^4-ia_1b_3\omega^5-ia_2\omega^3$$

$$+a_2b_1\omega^4+ia_2b_2\omega^5-a_2b_3\omega^6-a_3\omega^4-ia_3b_1\omega^5+a_3b_2\omega^6+ia_3b_3\omega^7$$

$$N3 = i(\omega-[a_2-a_1b_1+b_2]\omega^3-[a_1b_3-a_2b_2+a_3b_1]\omega^5+a_3b_3\omega^7) \quad \text{(III.C.6.11)}$$

$$+(a_1-b_1)\omega^2+(a_2b_1-a_1b_2-a_3+b_3)\omega^4+(a_3b_2-a_2b_3)\omega^6,$$

where a_1, a_2, a_3, b_1, b_2, and b_3 are defined in III.C.6.1 through III.C.6.6, respectively. Note from III.C.6.7 that all terms in the sum have the same denominator. Use III.C.6.10 and III.C.6.11 to determine the numerator of the second and

third terms and call it N. Then

$$N = i[-\omega+(a_2-a_1b_1+b_2)\omega^3+(a_1b_3-a_2b_2+a_3b_1)\omega^5-a_3b_3\omega^7+\omega-(a_2-a_1b_1+b_2)\omega^3$$

$$-(a_1b_3-a_2b_2+a_3b_1)\omega^5+a_3b_3\omega^7]+(a_1-b_1)\omega^2+(a_2b_1-a_1b_2-a_3+b_3)\omega^4$$

$$+(a_3b_2-a_2b_3)\omega^6+(a_1-b_1)\omega^2+(a_2b_1-a_1b_2-a_3+b_3)\omega^4+(a_3b_2-a_2b_3)\omega^6$$

$$= 2[(a_1-b_1)\omega^2+(a_2b_1-a_1b_2-a_3+b_2)\omega^4+(a_3b_2-a_2b_3)\omega^6].$$

Let

$$y = (a_1-b_1)+(a_2b_1-a_1b_2-a_3+b_3)\omega^2+(a_3b_2-a_2b_3)\omega^6]. \quad \text{(III.C.6.12)}$$

Then

$$N = 2\omega^2 y \quad \text{(III.C.6.13)}$$

Using III.C.6.7, III.C.6.8, III.C.6.13

$$g_+(\omega) = \frac{1}{\pi}\left(\frac{\omega^2 x}{\omega^2 x} + \frac{2\omega^2 y}{\omega^2 x}\right)$$

$$= \frac{1}{\pi}\left(\frac{x+2y}{x}\right), \quad \text{(III.C.6.14)}$$

where x and y are defined in III.C.6. 9 and III.C.6.12, respectively.

Figures III.C.6.1 through III.C.6.3 show the results of the calculation of the (Bartlett) spectrum of counts. In presenting the results the constant $\frac{1}{\pi}$ in III.C.6.14 was ignored. Figure III.C.6.1 shows the spectrum of counts for the same α and β values that were used for the sample paths and scatter plots of Figures III.C.3.1 through IIIC.3.6. This figure also shows the variation in the spectrum of counts as the $P(X_{n+1} > X_n)$ varies from its lowest to highest values. Figure III.C.3.2 holds the $P(X_{n+1} > X_n)$ constant and varies the correlation. Since the spectrum of counts for a Poisson process is a constant one when λ equals 1 and the constant $\frac{1}{\pi}$ is ignored, the correlation can be viewed as a measure of the process' departure from a Poisson process. This divergence as a function of the correlation shows clearly in this figure. Figure III.C.6.3 holds the correlation constant and varies the $P(X_{n+1} > X_n)$. The slight variation in the spectra shows that while the spectrum of counts does vary with the $P(X_{n+1} > X_n)$, the correlation plays a more dominant role.

The analysis from the Laplace transform of sums in III.C.4, through the Laplace transform of the intensity function in III.C.5, to the spectrum of counts in this section can be performed using the correlated $\{E_n\}$, $\{E_n'\}$ sequences of III.C.2 and thus for negative correlations. Details are not given.

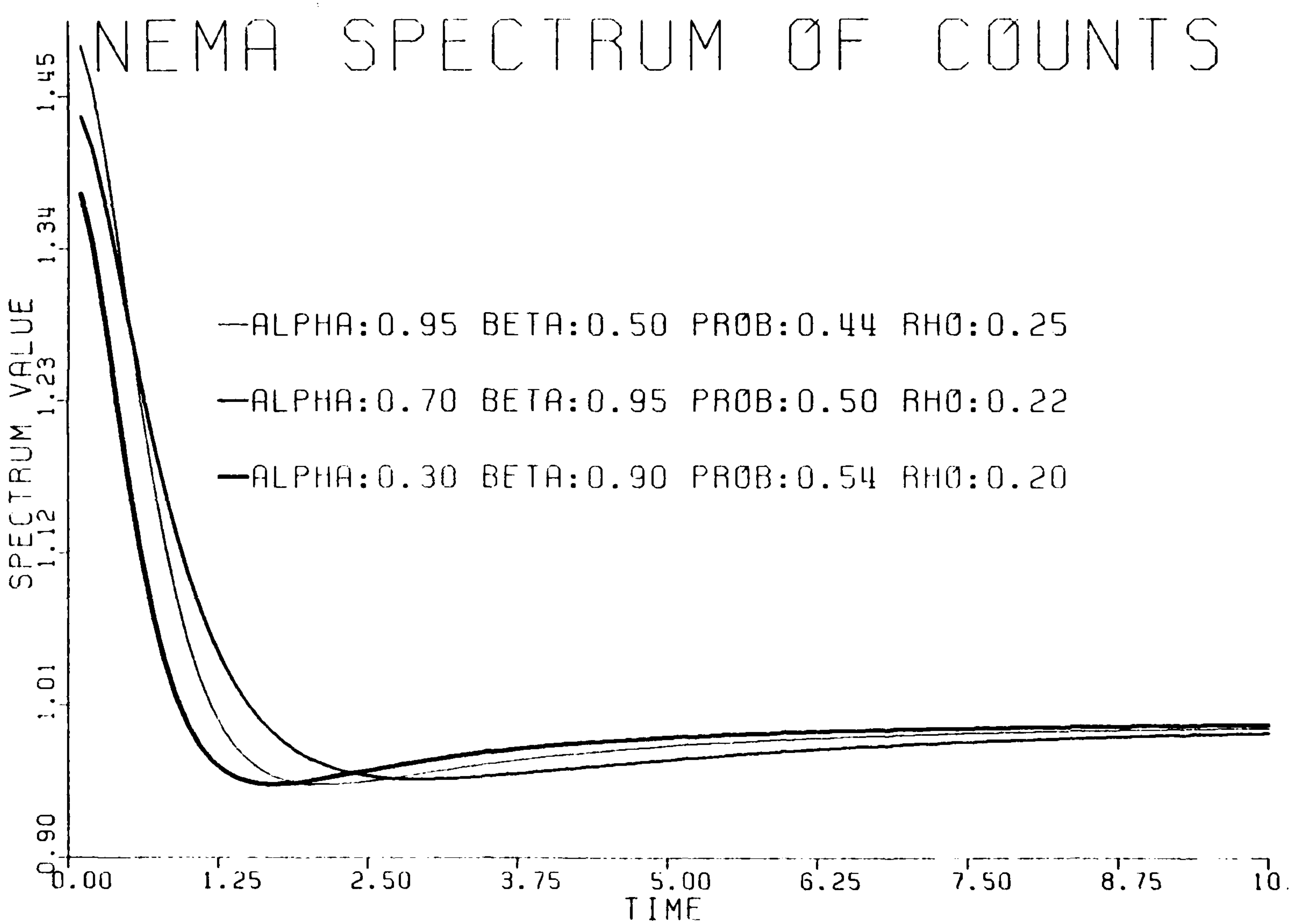


FIGURE III.C.6.1

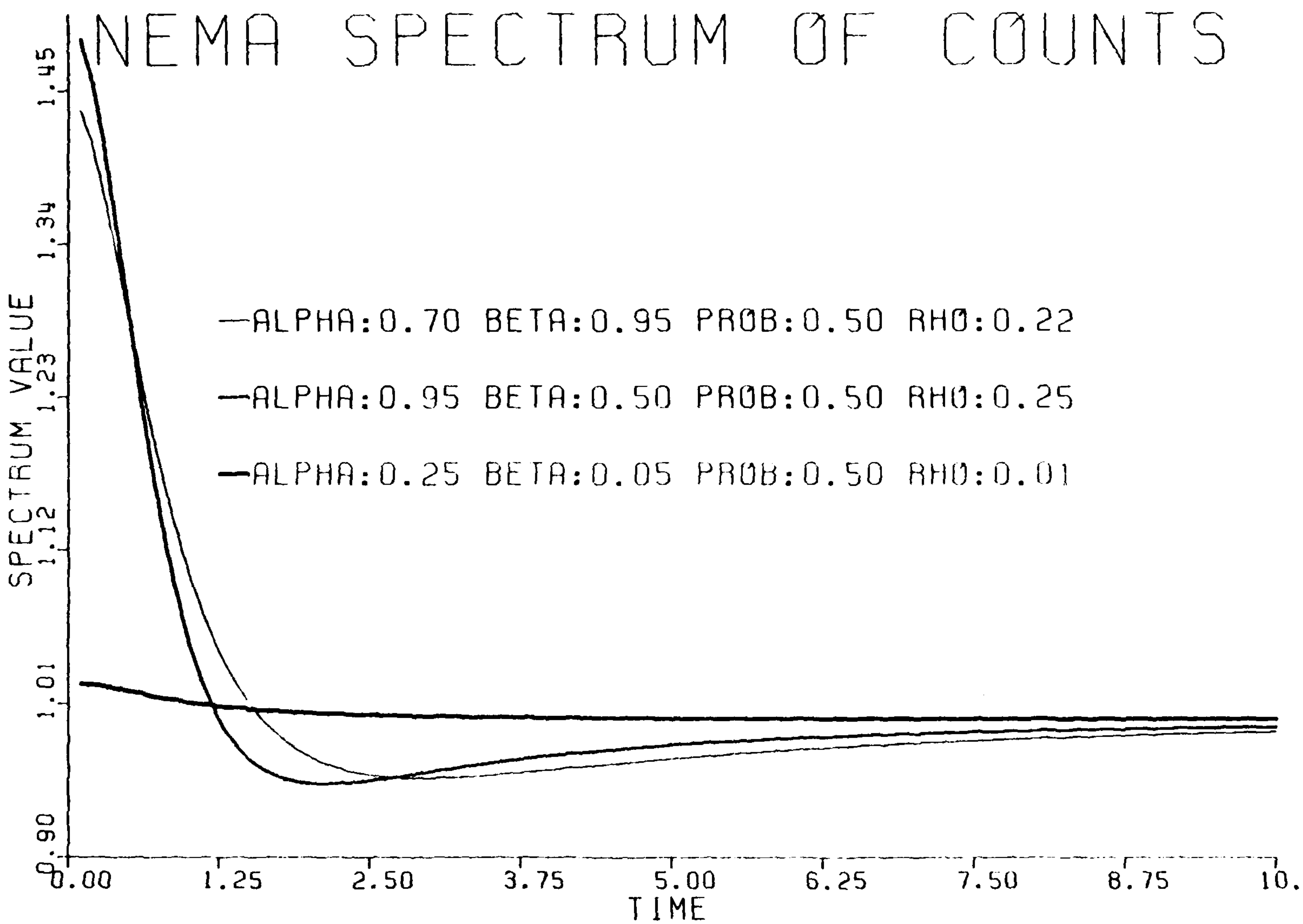


FIGURE III.C.6.2

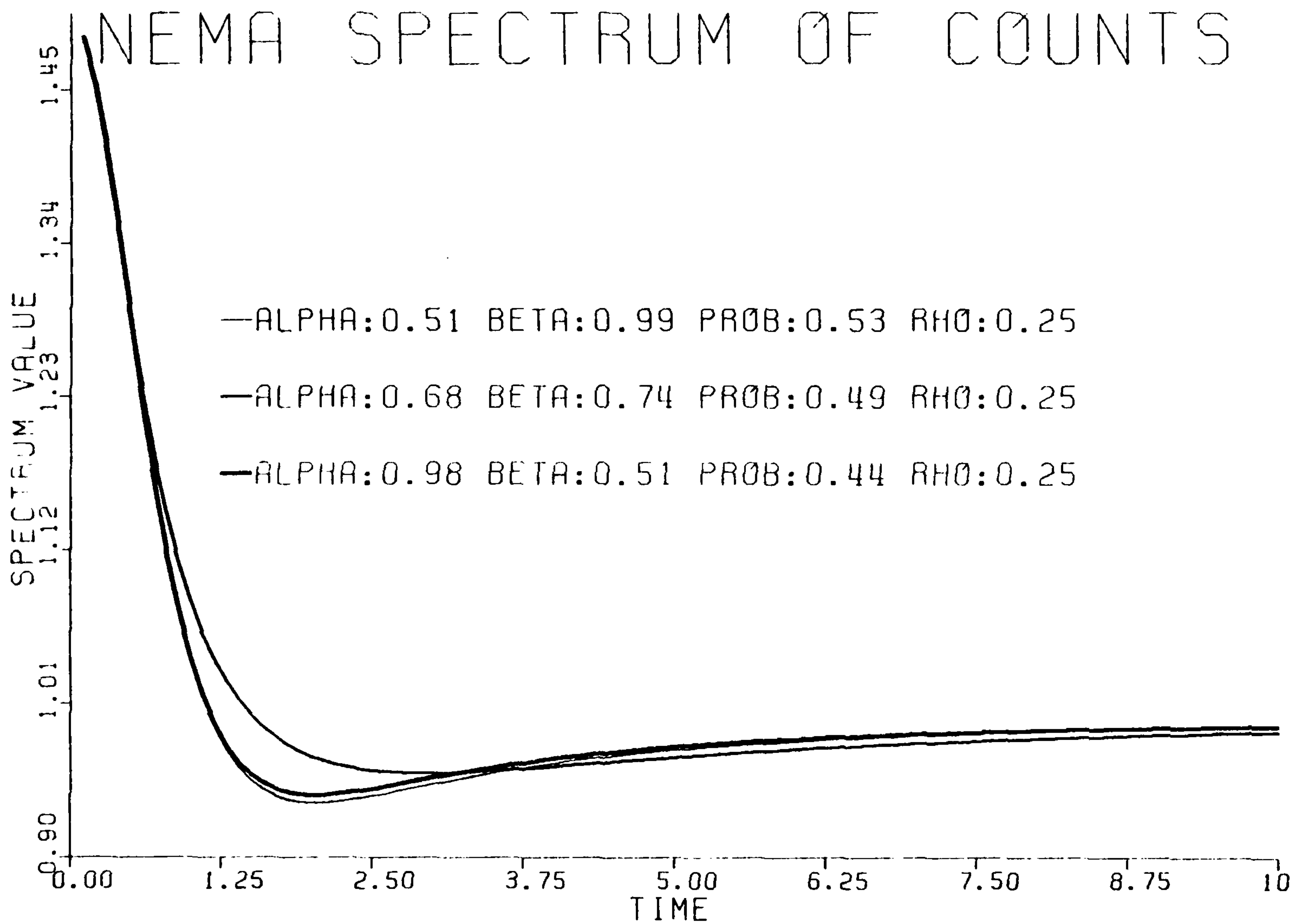


FIGURE III.C.6.3

7. Joint Laplace-Stieltjes Transform of X_n and X_{n+1}

Because the NEMA(1) process is, by construction, only one-dependent, all of the second-order properties of $\{X_n\}$ are contained in adjacent pairs $\{X_n, X_{n+1}\}$. In previous sections quantifiers of the distribution of $\{X_n, X_{n+1}\}$ such as ρ_j and $P(X_{n+1} > X_n)$ have been derived. Here we give the Laplace-Stieltjes transform of the joint distribution. One could, for example, study the effect of the two parameters from this result by deriving directional moments.

The joint Laplace-Stieltjes transform of X_n and X_{n+1} can be calculated by considering each of the sixteen possible combinations of parameter values for X_n and X_{n+1}, as was done in III.C.3. Let $\frac{1-\beta}{1-(1-\alpha)\beta} = \delta$, $(1-\alpha)\beta = \gamma$, $\phi_{X_n, X_{n+1}}(s_1, s_2) = E(e^{-s_1 X_n - s_2 X_{n+1}})$, and $\phi_E(s) = \frac{1}{1+s}$. Then

$$
\begin{aligned}
\Phi_{X_n,X_{n+1}}(s_1,s_2) &= \alpha\alpha\delta\delta E(e^{-s_1[E_n+\beta E_{n-1}]-s_2[E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)\alpha\delta\delta E(e^{-s_1[E_n+\beta E_{n-1}]-s_2 E_{n+1}}) \\
&+ \alpha(1-\alpha)\delta\delta E(e^{-s_1 E_n - s_2[E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)(1-\alpha)\delta\delta E(e^{-s_1 E_n - s_2 E_{n+1}}) \\
&+ \alpha\alpha\delta(1-\delta)E(e^{-s_1[\gamma E_n+\beta E_{n-1}]-s_2[E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)\alpha\delta(1-\delta)E(e^{-s_1[\gamma E_n+\beta E_{n-1}]-s_2 E_{n+1}}) \\
&+ \alpha(1-\alpha)\delta(1-\delta)E(e^{-s_1\gamma E_n - s_2[E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)(1-\alpha)\delta(1-\delta)E(e^{-s_1\gamma E_n - s_2 E_{n+1}}) \\
&+ \alpha\alpha(1-\delta)\delta E(e^{-s_1[E_n+\beta E_{n-1}]-s_2[\gamma E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)\alpha(1-\delta)\delta E(e^{-s_1[E_n+\beta E_{n-1}]-s_2\gamma E_{n+1}}) \\
&+ \alpha(1-\alpha)(1-\delta)\delta E(e^{-s_1 E_n - s_2[\gamma E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)(1-\alpha)(1-\delta)\delta E(e^{-s_1 E_n - s_2\gamma E_{n+1}}) \\
&+ \alpha\alpha(1-\delta)(1-\delta)E(e^{-s_1[\gamma E_n+\beta E_{n-1}]-s_2[\gamma E_{n+1}+\beta E_n]}) \\
&+ (1-\alpha)\alpha(1-\delta)(1-\delta)E(e^{-s_1[\gamma E_n+\beta E_{n-1}]-s_2\gamma E_{n+1}}) \\
&+ \alpha(1-\alpha)(1-\delta)(1-\delta)E(e^{-s_1\gamma E_n - s_2[\gamma E_{n-1}+\beta E_n]}) \\
&+ (1-\alpha)(1-\alpha)(1-\delta)(1-\delta)E(e^{-s_1\gamma E_n - s_2\gamma E_{n+1}})
\end{aligned}
$$

$$
\begin{aligned}
\phi_{X_n, X_{n+1}}(s_1, s_2) \quad = \quad & \alpha\alpha\delta\delta\phi_E(s_1+\beta s_2)\,\varepsilon_E(\beta s_1)\,\varepsilon_E(s_2) \\
& + (1-\alpha)\alpha\delta\delta\phi_E(s_1)\,\phi_E(\beta s_1)\,\phi_E(s_2) \\
& + \alpha(1-\alpha)\delta\delta\phi_E(s_1+\beta s_2)\,\phi_E(s_2) \\
& + (1-\alpha)(1-\alpha)\delta\delta\phi_E(s_1)\,\phi_E(s_2) \\
& + \alpha\alpha\delta(1-\delta)\,\phi_E(\gamma s_1+\beta s_2)\,\phi_E(\beta s_1)\,\phi_E(s_2) \\
& + (1-\alpha)\alpha\delta(1-\delta)\,\phi_E(\gamma s_1)\,\phi_E(\beta s_1)\,\phi_E(s_2) \\
& + \alpha(1-\alpha)\delta(1-\delta)\,\phi_E(\gamma s_1+\beta s_2)\,\phi_E(s_2) \\
& + (1-\alpha)(1-\alpha)\delta(1-\delta)\,\phi_E(\gamma s_1)\,\phi_E(s_2) \\
& + \alpha\alpha(1-\delta)\delta\phi_E(s_1+\beta s_2)\,\phi_E(\beta s_1)\,\phi_E(\gamma s_2) \\
& + (1-\alpha)\alpha(1-\delta)\delta\phi_E(s_1)\,\phi_E(\beta s_1)\,\phi_E(\gamma s_2) \\
& + \alpha(1-\alpha)(1-\delta)\delta\phi_E(s_1+\beta s_2)\,\phi_E(\gamma s_2) \\
& + (1-\alpha)(1-\alpha)(1-\delta)\delta\phi_E(s_1)\,\phi_E(\gamma s_2) \\
& + \alpha\alpha(1-\delta)(1-\delta)\,\phi_E(\gamma s_1+\beta s_2)\,\phi_E(\beta s_1)\,\phi_E(\gamma s_2) \\
& + (1-\alpha)\alpha(1-\delta)(1-\delta)\,\phi_E(\gamma s_1)\,\phi_E(\beta s_1)\,\phi_E(\gamma s_2) \\
& + \alpha(1-\alpha)(1-\delta)(1-\delta)\,\phi_E(\gamma s_1+\beta s_2)\,\phi_E(\gamma s_2) \\
& + (1-\alpha)(1-\alpha)(1-\delta)(1-\delta)\,\phi_E(\gamma s_1)\,\phi_E(\gamma s_2)
\end{aligned}
$$

$$\phi_{X_n,X_{n+1}}(s_1,s_2) = [\delta\phi_E(s_2)+(1-\delta)\phi_E(\gamma s_2)]$$

$$\times[\alpha\alpha\delta\phi_E(s_1+\beta s_2)\phi_E(\beta s_1) + (1-\alpha)\alpha\delta\phi_E(s_1)\phi_E(\beta s_1)$$

$$+ \alpha(1-\alpha)\delta\phi_E(s_1+\beta s_2) + (1-\alpha)(1-\alpha)\delta\phi_E(s_1)$$

$$+ \alpha\alpha(1-\delta)\phi_E(\gamma s_1+\beta s_2)\phi_E(\beta s_1)$$

$$+ (1-\alpha)\alpha(1-\delta)\phi_E(\gamma s_1)\phi_E(\beta s_1) + \alpha(1-\alpha)(1-\delta)\phi_E(\gamma s_1+\beta s_2)$$

$$+ (1-\alpha)(1-\alpha)(1-\delta)\phi_E(\gamma s_1)]$$

$$\phi_{X_n,X_{n+1}}(s_1,s_2) = [\delta\phi_E(s_2)+(1-\delta)\phi_E(\gamma s_2)] \tag{III.C.7.1}$$

$$\times[\alpha\phi_E(\beta s)+(1-\alpha)]\times[\alpha\delta\phi_E(s_1+\beta s_2)+(1-\alpha)\delta\phi_E(s_1)$$

$$+ \alpha(1-\delta)\phi_E(\gamma s_1+\beta s_2)+(1-\alpha)(1-\delta)\phi_E(\gamma s_1)]$$

For the special cases of the EMA(1) process, III.C.7.1 reduces to the results given in Lawrence and Lewis [Ref. 5].

D. THE MOVING MINIMUM MODEL

1. Introduction

Another possible scheme that can be used to generate one-dependent sequences of random variables with marginal Exponential distribution is the so-called minimum model. With this model the $\{X_n\}$ sequence is generated by taking the moving minimum value of two Exponential random variables. The proposed generation scheme is

$$X'_n = \mathrm{MIN}(E_n, bE_{n-1}), \tag{III.D.1.1}$$

where $\{X_n', n = 1,2,...\}$ is a sequence of random variables with marginal Exponential distribution, $\{E_n, n = 0,1,...\}$ is an iid sequence of Exponential random variables with unit mean, and $b \geq 0$. This will produce an $\{X_n'\}$ sequence with a rate of $\frac{b+1}{b}$ and an expected value of $\frac{b}{b+1}$. This expected value produces one difficulty since it is a function of the parameter, b. This complicates comparisons between results with different parameter values and decreases the value of scatter plots and sample paths. However, this difficulty can be easily removed by multiplying the $\{X_n'\}$ by $\frac{b+1}{b}$. The generation scheme then becomes

$$X_n = \frac{b+1}{b} X_n' = \text{MIN}([\tfrac{b+1}{b}]E_n, [b+1]E_{n-1}), \qquad \text{(III.D.1.2)}$$

with $\{E_n\}$ and b defined as before. The $\{X_n\}$ has a rate of one and, hence, an expected value of one. This facilitates comparisons for different parameter values with the NEMA(1) discussed in III.C which produces random variables with unit means.

The investigation of the moving minimum model is motivated by the previous result in III.C.2 that linear additive models have a constrained range of serial correlation. The hope is that the non-linearity of the moving minimum model will obviate this constraint. The minimum scheme has been used by Tavares [Refs. 22 and 30] to generate first-order autoregressive exponential processes and by Marshall and Olkin [Ref. 31] to generate correlated bivariate Exponential variables.

2. Correlation Structure

The first-order serial correlation can be computed by the following approach

$$E(X_{n+1}X_n) = E([MIN([\tfrac{b+1}{b}]E_{n+1},[b+1]E_n)]$$
$$[MIN([\tfrac{b+1}{b}]E_n,[b+1]E_{n-1})]).$$

The terms inside the expected value can be made independent by conditioning on the value of E_n. The $E(X_{n+1}X_n)$ is then found by multiplying the conditional result by the density of E_n, and integrating. Implementing this approach we have

$$E(X_{n+1}X_n) = \int_0^{\infty} E([MIN([\tfrac{b+1}{b}]E_{n+1},[b+1]y)]$$
$$\times[MIN([\tfrac{b+1}{b}]y,[b+1]E_{n-1}] \mid E_{n=y})e^{-y}dy$$

$$E(X_{n+1}X_n) = \int_0^{\infty}(E[MIN([\tfrac{b+1}{b}]E_{n+1},[b+1]y)]) \qquad \text{(III.D.2.1)}$$
$$\times(E[MIN([\tfrac{b+1}{b}]y,[b+1]E_{n-1})])e^{-y}dy$$

The expected value of the minima can be calculated as follows:

$$E(MIN[(\tfrac{b+1}{b})E_{n+1},(b+1)y]) = \int_0^{(b+1)y} x(\tfrac{b}{b+1})e^{-\frac{bx}{b+1}}dx$$
$$+\int_{(b+1)y}^{\infty}(b+1)y(\tfrac{b}{b+1})e^{-\frac{bx}{b+1}}dx$$

$$E(MIN[(\frac{b+1}{b})E_{n+1},(b+1)y]) = -xe^{-\frac{bx}{b+1}}\Big|_0^{(b+1)y}$$

$$+ \int_0^{(b+1)y} e^{-\frac{bx}{b+1}}dx + (b+1)ye^{-by}$$

$$= -(b+1)ye^{-by} + (\frac{b+1}{b})\int_0^{(b+1)y} (\frac{b}{b+1})e^{-\frac{bx}{b+1}}dx$$

$$+ (b+1)ye^{-by}$$

$$E(MIN[(\frac{b+1}{b})E_{n+1},(b+1)y]) = \frac{b+1}{b}(1 - e^{-by}). \qquad (III.D.2.2)$$

Similarly,

$$E(MIN[(\frac{b+1}{b})y,(b+1)E_{n-1}]) = \int_0^{(b+1)y/b} x(\frac{1}{b+1})e^{-\frac{x}{b+1}}dx$$

$$+ \int_{(b+1)y/b}^{\infty} \frac{(b+1)y}{b}(\frac{1}{b+1})e^{-\frac{x}{b+1}}dx$$

$$= -xe^{-\frac{x}{b+1}}\Big|_0^{(b+1)y/b} + \int_0^{(b+1)y/b} e^{-\frac{x}{b+1}}dx$$

$$+ \frac{(b+1)y}{b} e^{-y/b}$$

$$= -\frac{(b+1)y}{b} e^{-y/b} + (b+1)\int_0^{(b+1)y/b} (\frac{1}{b+1})e^{-\frac{x}{b+1}}dx$$

$$+ \frac{(b+1)y}{b} e^{-y/b}$$

$$E(\mathrm{MIN}[(\tfrac{b+1}{b})y,(b+1)E_{n-1}]) = (b+1)(1-e^{-y/b}). \qquad \text{(III.D.2.3)}$$

Using III.D.2.2 and III.D.2.3 in III.D.2.1 produces

$$E(X_{n+1}X_n) = \int_0^\infty (\tfrac{b+1}{b})(1-e^{-by})(b+1)(1-e^{-y/b})e^{-y}dy$$

$$= \frac{(b+1)^2}{b} - \frac{b+1}{b} - (b+1) + \frac{(b+1)^2}{b(1+b+\frac{1}{b})}$$

$$= \frac{(b+1)^2}{b^2+b+1}.$$

Therefore, since $E(X_n) = 1$

$$\mathrm{COV}(X_{n+1},X_n) = \frac{(b+1)^2}{b^2+b+1} - 1 = \frac{b}{b^2+b+1}$$

and

$$\mathrm{CORR}(X_{n+1},X_n) = \frac{b}{b^2+b+1} \qquad \text{(III.D.2.4)}$$

Thus the model allows a range of correlations from $[0,\frac{1}{3}]$. The minimum value is achieved when b is zero or in the limit as b tends to infinity. The maximum value is achieved when b is equal to one. An interesting aspect of the correlation structure of the moving minimum model is that reciprocal values of b produce equal correlations. This is a similar kind of "invertibility" found for the other moving average models discussed in III.C. The range of b could be restricted

to the interval [0,1] without reducing the possible range of correlations. However, doing so, as with the NEMA(1) model, would ignore the fact that in the non-normal case characteristics other than correlation may be different in the case of b and $\frac{1}{b}$. Also, as is the case for all the first-order moving average processes addressed in this paper, the correlation for lags greater than one are zero. So the study of correlation structure is limited to the study of the serial correlation with lag one.

3. Negative Correlation

The range of possible correlations can be extended in a fashion similar to the NEMA(1) model (see III.C.2) by the use of correlated or antithetic variables. Using this approach the generation formula becomes for antithetic variables

$$X_n = \mathrm{MIN}([\frac{b+1}{b}]E_n, [b+1]E'_{n-1}), \tag{III.D.3.1}$$

where all variables are defined as in III.D.1.1 and $\{E'_n, n = 0,1,...\}$ is generated from the $\{E_n\}$ sequence using the relationship $E'_n = -\ln(1-e^{-E_n})$. Note that this implies that $\{E'_n\}$ is also iid Exponential with unit mean. Again

$$E(X_nX_{n+1}) = E[(\mathrm{MIN}[(\frac{b+1}{b})E_n, (b+1)E'_{n-1}]) \times (\mathrm{MIN}[(\frac{b+1}{b})E_{n+1}, (b+1)E'_n])]$$

and conditioning on the value of E_n, multiplying by the density

of E_n, and integrating produces

$$E(X_nX_{n+1}) = \int_0^\infty E([MIN([\tfrac{b+1}{b}]y,[b+1]E'_{n-1})]$$

$$[MIN([\tfrac{b+1}{b}]E_{n+1},[b+1]E'_n)] \mid E_n = y)e^{-y}dy$$

$$E(X_nX_{n+1}) = \int_0^\infty (E[MIN([\tfrac{b+1}{b}]y,[b+1]E'_{n-1})]) \qquad \text{(III.D.3.2)}$$

$$(E[MIN([\tfrac{b+1}{b}]E_{n+1},-[b+1]\ln[1-e^{-y}])])e^{-y}dy$$

The first expected value is identical to III.D.2.3. Thus

$$E(MIN[(\tfrac{b+1}{b})y,(b+1)E'_{n-1}]) = (b+1)(1-e^{-y/b}) \qquad \text{(III.D.3.3)}$$

The second can be calculated as before.

$$E(MIN[(\tfrac{b+1}{b})E_{n+1},-(b+1)\ln(1-e^{-y})])$$

$$= \int_0^{-(b+1)\ln(1-e^{-y})} x(\tfrac{b}{b+1})e^{-\frac{bx}{b+1}}dx$$

$$+ \int_{-(b+1)\ln(1-e^{-y})}^\infty -(b+1)\ln(1-e^{-y})e^{-\frac{bx}{b+1}}dx$$

$$= -xe^{-\frac{bx}{b+1}}\Big|_0^{-(b+1)\ln(1-e^{-y})} + (\tfrac{b+1}{b})\int_0^{-(b+1)\ln(1-e^{-y})}(\tfrac{b}{b+1})e^{-\frac{bx}{b+1}}dx$$

$$- (b+1)\ln(1-e^{-y})e^{b\ln(1-e^{-y})}$$

$$E(\mathrm{MIN}[(\tfrac{b+1}{b})E_{n+1}, -(b+1)\ln(1-e^{-y})])$$

$$= (b+1)\ln(1-e^{-y})e^{b\ln(1-e^{-y})} + (\tfrac{b+1}{b})(1-e^{b\ln[1-e^{-y}]})$$

$$- (b+1)\ln(1-e^{-y})e^{b\ln(1-e^{-y})}$$

$$E(\mathrm{MIN}[(\tfrac{b+1}{b})E_{n+1}, -(b+1)\ln(1-e^{-y})])$$

$$= (\tfrac{b+1}{b})(1 - [1-e^{-y}]^b) \qquad \text{(III.D.3.4)}$$

Substituting III.D.3.3 and III.D.3.4 into III.D.3.2 yields

$$E(X_nX_{n+1}) = \int_0^\infty (\tfrac{b+1}{b})(1-e^{-y/b})(\tfrac{b+1}{b})(1-[1-e^{-y}]^b)e^{-y}dy$$

$$= (\tfrac{b+1}{b})^2\int_0^\infty e^{-y}dy - (\tfrac{b+1}{b})\int_0^\infty (\tfrac{b+1}{b})e^{-(\frac{b+1}{b})y}dy$$

$$- (\tfrac{b+1}{b})^2\int_0^\infty e^{-y}(1-e^{-y})^b dy$$

$$+ (\tfrac{b+1}{b})^2\int_0^\infty e^{-(1+\frac{1}{b})y}(1-e^{-y})^b dy$$

The first two integrals are trivial. In the third the change of variable $z = (1-e^{-y})$, $dz = e^{-y}dy$ makes that integral straightforward. In the last integral, the change of variable $u = e^{-y}$, $-\frac{du}{u} = dy$ makes that integral recognizable as the integral of a Beta random variable. Using these changes of variables and

making the appropriate changes to the limits of integration produces

$$E(X_n X_{n+1}) = \left(\frac{b+1}{b}\right)^2 - \frac{b+1}{b} - \left(\frac{b+1}{b}\right)^2 \int_0^1 z^b dz + \int_b^0 u^{-(1+\frac{1}{b})} (1-u)^b \left(-\frac{du}{u}\right)$$

$$= \left(\frac{b+1}{b}\right) - \left(\frac{b+1}{b}\right) - \frac{b+1}{b^2} + \int_0^1 u^{1/b} (1-u)^b du$$

$$= \frac{\Gamma(1+\frac{1}{b})\Gamma(1+b)}{\Gamma(2+b+\frac{1}{b})}$$

$$= \frac{\frac{1}{b}\Gamma(\frac{1}{b}) b\Gamma(b)}{(1+b+\frac{1}{b})(b+\frac{1}{b})\Gamma(b+\frac{1}{b})}$$

$$= \frac{b^2\Gamma(\frac{1}{b})\Gamma(b)}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})}.$$

Then

$$COV(X_n, X_{n+1}) = \frac{b^2\Gamma(\frac{1}{b})\Gamma(b)}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} - 1$$

and

$$CORR(X_n, X_{n+1}) = \frac{b^2\Gamma(\frac{1}{b})\Gamma(b)}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} - 1 \qquad \text{(III.D.3.5)}$$

Like the expression for positive correlation, this expression is also symmetric with respect to reciprocal values of the parameter. It attains a minimum value of minus one-third when the parameter value is one. Graphs of the

correlation as a function of the parameter b for both positive and negative correlations are provided in Figures III.D.3.1 and III.D.3.2, respectively. Unlike the NEMA(1) model which requires an additional parameter (see equation III.B.6) to achieve a full range of negative correlations, the moving minimum model can achieve its full range with the single parameter b and an antithetic sequence $\{E_n'\}$.

4. Joint Density of X_n and X_{n+1}

Calculation of the joint density of X_n and X_{n+1} is possible using a conditioning argument to determine $P(X_n \leq x | E_n = z)$ and $P(X_{n+1} \leq y | E_n = z)$. These values along with the probability that E_n takes on a given range of values are sufficient to determine the joint distribution function of X_n and X_{n+1}. The form of the distribution will vary depending on whether one is above or below the line $X_{n+1} = bX_n$. The joint density, where it exists, is determined by differentiating the distribution function.

From III.D.1.2 we have

$$X_n = \text{MIN}([\tfrac{b+1}{b}]E_n, [b+1]E_{n-1}).$$

Then

$$P(X_n \leq x | E_n = z) = \begin{cases} 1 & \text{if } (\frac{b+1}{b})z \leq x, \\ 1-e^{-\frac{x}{b+1}} & \text{if } (\frac{b+1}{b})z > x. \end{cases} \qquad \text{(III.D.4.1)}$$

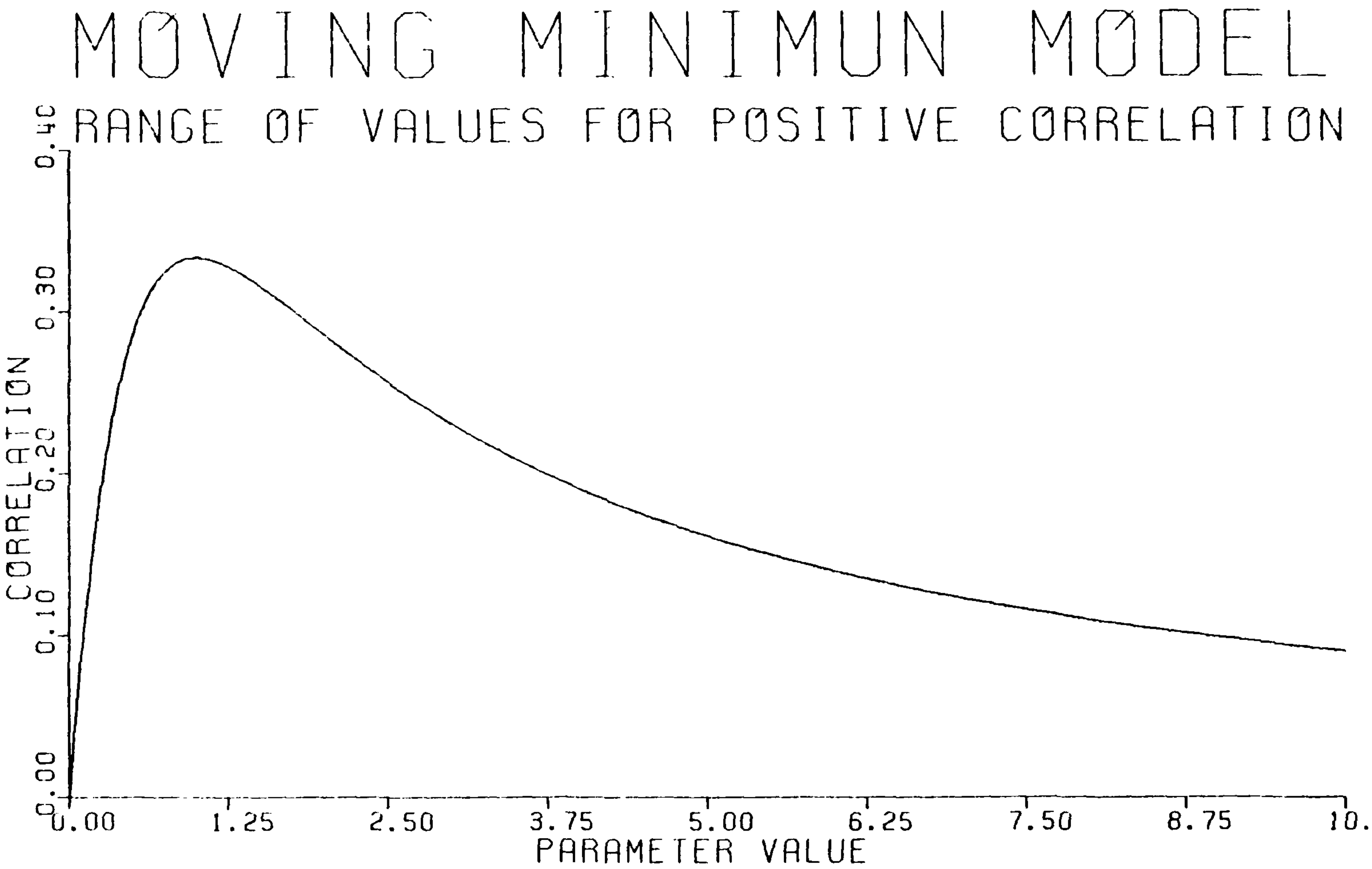


FIGURE III.D.3.1

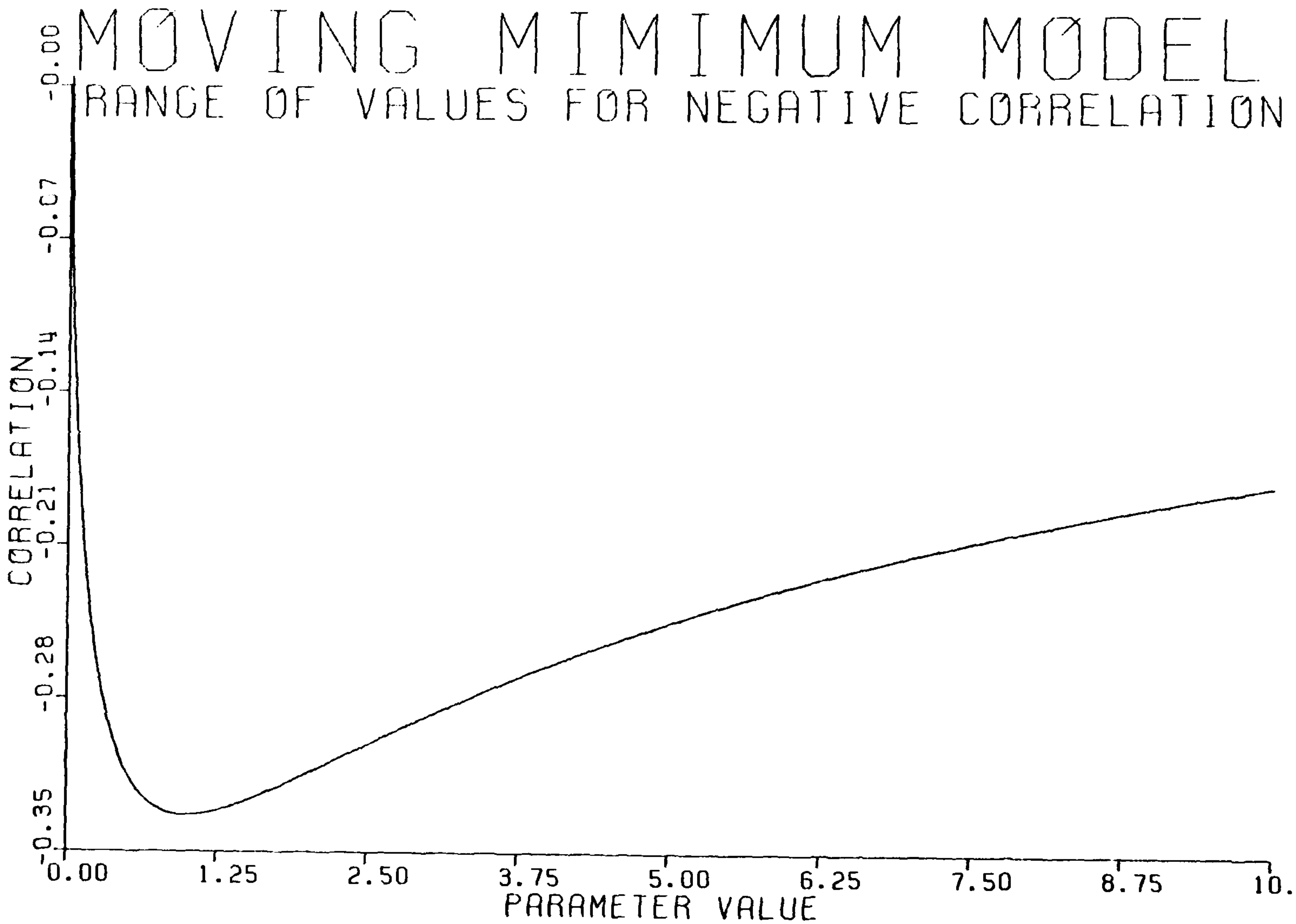


FIGURE III.D.3.2

The first result in the above is obvious. To justify the second, consider

$$P(X_n \le x \mid E_n = z > \frac{bx}{b+1}) = P([b+1]E_{n-1} \le x)$$

$$= P(E_{n-1} \le \frac{x}{b+1})$$

$$= 1 - e^{-\frac{x}{b+1}}.$$

Since $X_{n+1} = \text{MIN}([\frac{b+1}{b}]E_{n+1}, [b+1]E_n)$, then

$$P(X_{n+1} \le y \mid E_n = z) = \begin{cases} 1 & \text{if} \quad z \le \frac{y}{b+1}, \\ 1-e^{-\frac{by}{b+1}} & \text{if} \quad z > \frac{y}{b+1}. \end{cases} \qquad \text{(III.D.4.2)}$$

To consider the joint distribution, note that when $\frac{xb}{b+1} < \frac{y}{b+1}$ (i.e. when you are above the line $bx = y$), the range of possible z values can be broken up into three regions. See Figure III.D.4.1. Then, since E_n is Exponential with unit mean,

$$P(z \in \text{REGION 1}) = P(z \le \frac{bx}{b+1}),$$

$$P(z \in \text{REGION 1}) = 1 - e^{-\frac{bx}{b+1}}. \qquad \text{(III.D.4.3)}$$

$$P(z \in \text{REGION 2}) = P(\frac{bx}{b+1} < z \le \frac{y}{b+1}),$$

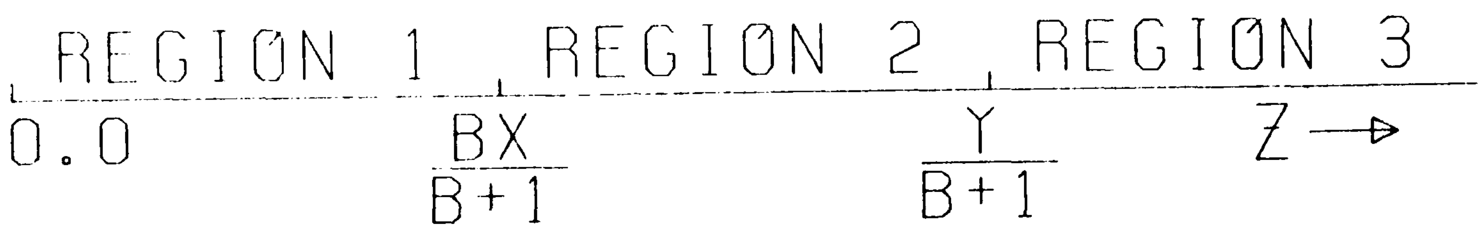


FIGURE III.D.4.1

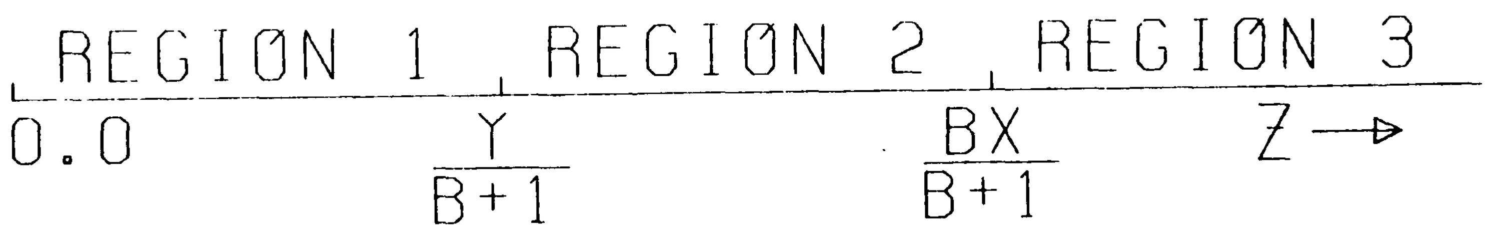


FIGURE III.D.4.2

$$P(z \in \text{REGION } 2) = e^{-\frac{bx}{b+1}} - e^{-\frac{y}{b+1}}. \qquad \text{(III.D.4.4)}$$

$$P(z \in \text{REGION } 3) = P(z > \frac{y}{b+1})$$

$$P(z \in \text{REGION } 3) = e^{-\frac{y}{b+1}} \qquad \text{(III.D.4.5)}$$

Now by definition

$$P(X_n \leq X, X_{n+1} \leq y | z \in \text{REGION } i) = P(X_n \leq x | z \in \text{REGION } i)$$

$$\times P(X_{n+1} \leq y | z \in \text{REGION } i) \qquad \text{(III.C.4.6)}$$

because when conditioned on the value of E_n, these probabilities are independent. Using the above equation, III.D.4.1, III.D.4.2, and the definition of the regions in Figure III.D.4.1,

$$P(X_n \leq x, X_{n+1} \leq y | z \in \text{REGION } 1) = 1 \qquad \text{(III.D.4.7)}$$

$$P(X_n \leq x, X_{n+1} \leq y | z \in \text{REGION } 2) = 1 - e^{-\frac{x}{b+1}} \qquad \text{(III.D.4.8)}$$

$$P(X_n \leq x, X_{n+1} \leq y | z \in \text{REGION } 3) = (1-e^{-\frac{x}{b+1}})(1-e^{-\frac{by}{b+1}}) \qquad \text{(III.D.4.9)}$$

Using the results of III.D.4.3 through III.D.4.5 and III.D.4.7 through III.D.4.9 we can compute the joint distribution of X_n and X_{n+1} when $y > bx$ by using the relation

$$P(X_n \le x, X_{n+1} \le y) = \sum_{i=1}^{3} P(X_n \le x, X_{n+1} \le y | z \in \text{REGION } i) \times P(z \in \text{REGION } i) \qquad \text{(III.D.4.10)}$$

$$= 1(1-e^{-\frac{bx}{b+1}}) + (1-e^{-\frac{x}{b+1}})(e^{-\frac{bx}{b+1}} - e^{-y/b+1})$$

$$+ (1-e^{-\frac{x}{b+1}})(1-e^{-\frac{by}{b+1}})e^{-y/(b+1)}$$

$$= 1-e^{-\frac{bx}{b+1}}+e^{-\frac{bx}{b+1}}-e^{-x}-e^{-\frac{y}{b+1}}+e^{-\frac{(x+y)}{b+1}}+e^{-\frac{y}{b+1}}$$

$$- e^{-\frac{(x+y)}{b+1}}-e^{-y}+e^{-(\frac{1}{b+1})(x+[b+1]y)}$$

$$P(X_n \le x, X_{n+1} \le y) = 1-e^{-x}-e^{-y}+e^{-\frac{x}{b+1}-y} \qquad \text{(III.D.4.11)}$$

Similarly, when $y < bx$ (i.e. when you are below the line $bx = y$), the range of possible z values can be separated into three regions. See Figure III.D.4.2. Then

$$P(z \in \text{REGION 1}) = P(z \le \frac{y}{b+1}),$$

$$P(z \in \text{REGION 2}) = 1 - e^{-\frac{y}{b+1}}, \qquad \text{(III.D.4.12)}$$

$$P(z \in \text{REGION 2}) = P(\frac{y}{b+1} < z \le \frac{bx}{b+1}),$$

$$P(z \in \text{REGION 2}) = e^{-\frac{y}{b+1}} - e^{-\frac{bx}{b+1}}. \qquad \text{(III.D.4.13)}$$

$$P(z \in \text{REGION 3}) = P(z > \frac{bx}{b+1})$$

$$P(z \in \text{REGION } 3) = e^{-\frac{bx}{b+1}} \qquad \text{(III.D.4.14)}$$

Using III.D.4.1, III.D.4.2, III.D.4.6, and the definitions of the regions in Figure III.D.4.2, the following results hold for the given region.

$$P(X_n \le x, X_{n+1} \le y \mid z \in \text{REGION } 1) = 1 \qquad \text{(III.D.4.15)}$$

$$P(X_n \le x, X_{n+1} \le y \mid z \in \text{REGION } 2) = 1 - e^{-\frac{by}{b+1}} \qquad \text{(III.D.4.16)}$$

$$P(X_n \le x, X_{n+1} \le y \mid z \in \text{REGION } 3) = (1-e^{-\frac{by}{b+1}}) \times (1-e^{-\frac{x}{b+1}}) \qquad \text{(III.D.4.17)}$$

Combining III.D.4.10 with III.D.4.12 through III.D.4.17 yields for $bx < y$

$$P(X_n \le x, X_{n+1} \le y) = 1-e^{-y}-e^{-x}+e^{-x-\frac{by}{b+1}} \qquad \text{(III.D.4.18)}$$

Let $F^1(x,y) = 1-e^{-x}-e^{-y}+e^{-\frac{x}{b+1}-y}$, the distribution function of X_n and X_{n+1} when $bx < y$; and let $f^1(x,y)$ be the joint density of X_n and X_{n+1} when $bx < y$. Then

$$f^1(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F^1(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}[1-e^{-x}-e^{-y}+e^{-\frac{x}{b+1}-y}]$$

$$= \frac{\partial}{\partial x}[e^{-y}-e^{-\frac{x}{b+1}-y}]$$

$$f^1(x,y) = \left(\frac{1}{b+1}\right)e^{-\frac{x}{b+1}-y} \qquad bx < y; \quad x > 0. \qquad \text{(III.D.4.19)}$$

For $bx < y$, the distribution function, $F^2(x,y)$, is $1-e^{-y}-e^{-x}+e^{-x-\frac{by}{b+1}}$. If $f^2(x,y)$ is the joint density of X_n and X_{n+1} when $bx > y$, then

$$f^2(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F^2(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}[1-e^{-y}-e^{-x}+e^{-x-\frac{by}{b+1}}]$$

$$= \frac{\partial}{\partial x}[e^{-y}-\left(\frac{b}{b+1}\right)e^{-x-\frac{by}{b+1}}]$$

$$f^2(x,y) = \left(\frac{b}{b+1}\right)e^{-x-\frac{by}{b+1}} \qquad y < bx; \quad y > 0. \qquad \text{(III.D.4.20)}$$

Note that there is a positive probability that the point (X_n, X_{n+1}) lies on the line $bx = y$. This probability can be computed as follows. We have

$$X_n = \text{MIN}([\tfrac{b+1}{b}]E_n, [b+1]E_{n-1})$$

$$X_{n+1} = \text{MIN}([\tfrac{b+1}{b}]E_{n+1}, [b+1]E_n)$$

The point (X_n, X_{n+1}) lies on the line $bx = y$ when $X_n = (\frac{b+1}{b})E_n$ and $X_{n+1} = (b+1)E_n$. Now

$$P(X_n = [\tfrac{b+1}{b}]E_n; X_{n+1} = [b+1]E_n)$$

$$= P([\tfrac{b+1}{b}]E_n \le [b+1]E_{n-1}; [b+1]E_n \le [\tfrac{b+1}{b}]E_{n+1})$$

The events in the right hand side can be made independent by conditioning on the value of E_n. Then

$$P(X_n = [\tfrac{b+1}{b}]E_n; X_{n+1} = [b+1]E_n)$$

$$= \int_0^\infty P([\tfrac{b+1}{b}]y \leq [b+1]E_{n-1}, [b+1]y \leq [\tfrac{b+1}{b}]E_{n+1} \mid E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E_{n-1} > \tfrac{y}{b})P(E_{n+1} > by)e^{-y}dy$$

$$= \frac{1}{b+1+\frac{1}{b}} \int_0^\infty (b+1+\tfrac{1}{b})e^{-(b+1+\frac{1}{b})y}dy$$

$$P(X_n = [\tfrac{b+1}{b}]E_n; X_{n+1} = [b+1]E_n) = \frac{b}{b^2+b+1}. \qquad \text{(III.D.4.21)}$$

Because there is a positive probability of lying on the line $bx = y$, the moving minimum model can be said to have a line degeneracy. An important implication of the positive probability of (X_n, X_{n+1}) lying on the line $bx = y$ is that the moving minimum model will produce runs of values of constant ratio b. The values of $\{X_n\}$ in these runs will be decreasing, equal, or increasing for b less than, equal to, or greater than one, respectively. The length of the runs will be geometrically distributed with parameter $\frac{b}{b^2+b+1}$ for the positive correlation case. It was this kind of degeneracy in the Exponential autoregressive model, EAR(1), that proved to be one of the model's major weaknesses. The degeneracy also

occurs in the Tavares autoregressive model and the bivariate Exponential pairs derived by Marshall and Olkin.

The probability of lying above or below the line $bx = y$ can be easily found by integrating the appropriate joint density over the area desired. Thus, for $bx < y$,

$$P(\text{lying above } bx=y) = \int_0^\infty \int_{bx}^\infty f^1(x,y)\,dydx$$

$$= \int_0^\infty \int_{bx}^\infty \left(\frac{1}{b+1}\right) e^{-\frac{x}{b+1}-y}\,dydx$$

$$= \int_0^\infty \left(\frac{1}{b+1}\right) e^{-\frac{x}{b+1}} e^{-bx}\,dx$$

$$P(\text{lying above } bx=y) = \frac{1}{b^2+b+1} \qquad \text{(III.D.4.22)}$$

Similarly for $bx > y$

$$P(\text{lying below } bx=y) = \int_0^\infty \int_0^{bx} f^2(x,y)\,dydx$$

$$= \int_0^\infty \int_0^{bx} \left(\frac{b}{b+1}\right) e^{-x-\frac{by}{b+1}}\,dydx$$

$$= \int_0^\infty e^{-x}\left(1-e^{-\frac{b^2x}{b+1}}\right)dx$$

$$= 1 - \frac{b+1}{b^2+b+1}$$

$$P(\text{lying below } bx=y) = \frac{b^2}{b^2+b+1} \qquad \text{(III.D.4.23)}$$

5. Conditional Expectation and $P(X_{n+1} > X_n)$

Besides the correlation coefficient, there are two other characterizations of the joint distribution of X_n and X_{n+1} which we have considered. They are the conditional expectations and the $P(X_{n+1} > X_n)$. Both of these quantities can be derived by considering the four possible sets of values for X_n and X_{n+1}, computing the probability of each set occurring, and weighting the conditional expectation or probability by its probability of occurrence.

First, the probability of occurrence for each set of values must be calculated. Consider the case where $X_n = (\frac{b+1}{b})E_n$ and $X_{n+1} = (\frac{b+1}{b})E_{n+1}$.

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= P([\tfrac{b+1}{b}]E_n \leq [b+1]E_{n-1}, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]E_n)$$

By conditioning on the value of E_n, the events on the RHS become independent. The calculation then proceeds in a straightforward way.

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \int_0^\infty P([\tfrac{b+1}{b}]y \leq [b+1]E_{n-1}, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]y \mid E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E_{n-1} > \tfrac{y}{b})P(E_{n+1} \leq by)e^{-y}dy$$

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \int_0^\infty e^{-(\frac{1}{b}+1)y}\,dy - \int_0^\infty e^{-(b+1+\frac{1}{b})y}\,dy$$

$$= \frac{b}{b+1} - \frac{b}{b^2+b+1}$$

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1}) = \frac{b^3}{(b+1)(b^2+b+1)} \qquad \text{(III.D.5.1)}$$

The second case when $X_n = (\frac{b+1}{b})E_n$ and $X_{n+1} = (b+1)E_n$ has already been computed. The result is III.D.4.21 and is repeated here as

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n) = \frac{b}{b^2+b+1} \qquad \text{(III.D.5.2)}$$

The third case is when $X_n = (b+1)E_{n-1}$ and $X_{n+1} = (\frac{b+1}{b})E_{n+1}$. Here we proceed as in the first case.

$$P(X_n = [b+1]E_{n-1}, X_n = [\tfrac{b+1}{b}]E_{n+1})$$

$$= P([b+1]E_{n-1} \leq [\tfrac{b+1}{b}]E_n, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]E_n)$$

$$= \int_0^\infty P([b+1]E_{n-1} \leq [\tfrac{b+1}{b}]y, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]y \mid E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E_{n-1} \leq \tfrac{y}{b})P(E_{n+1} \leq by)e^{-y}dy$$

$$P(X_n = [b+1]E_{n-1}, X_n = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \int_0^\infty e^{-y}dy - \int_0^\infty e^{-(1+\frac{1}{b})y}dy - \int_0^\infty e^{-(b+1)y}dy + \int_0^\infty e^{-(b+1+\frac{1}{b})y}dy$$

$$= 1 - \frac{b}{b+1} - \frac{1}{b+1} + \frac{1}{b^2+b+1}$$

$$P(X_n = [b+1]E_{n-1}, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1}) = \frac{b}{b^2+b+1} \qquad \text{(III.D.5.3)}$$

The final case is when $X_n = (b+1)E_{n-1}$ and $X_{n+1} = (b+1)E_n$. As before

$$P(X_n = [b+1]E_{n-1}, X_{n+1} = [b+1]E_n)$$

$$= \int_0^\infty P([b+1]E_{n-1} \leq [\tfrac{b+1}{b}]y, [b+1]y \leq [\tfrac{b+1}{b}]E_{n+1} | E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E_{n-1} \leq \tfrac{y}{b})P(E_{n+1} > by)e^{-y}dy$$

$$= \int_0^\infty e^{-(b+1)y}dy - \int_0^\infty e^{-(b+1+\frac{1}{b})y}dy$$

$$= \frac{1}{b+1} - \frac{b}{b^2+b+1}$$

$$P(X_n = [b+1]E_{n-1}, X_{n+1} = [b+1]E_n) = \frac{1}{(b+1)(b^2+b+1)} \qquad \text{(III.D.5.4)}$$

The conditional expectations can now be written by inspection.

Case	$E(X_{n+1} \mid X_n = y)$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{b+1}{b}$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n$	by
$X_n = [b+1]E_{n-1}, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{b+1}{b}$
$X_n = [b+1]E_{n-1}, X_{n+1} = [b+1]E_{n-1}$	$b+1$

Weighting these conditional expectations by the probabilities in III.D.5.1 through III.D.5.4 yields the final result.

$$E(X_{n+1} \mid X_n = y) = \sum_{i=1}^{4} E(X_{n+1} \mid X_n = y; \text{case } i) P(\text{case } i)$$

$$= [\frac{b+1}{b}][\frac{b^3}{(b+1)(b^2+b+1)}] + (by)(\frac{b}{b^2+b+1}) + (\frac{b+1}{b})(\frac{b}{b^2+b+1}) + (b+1)(\frac{1}{[b+1][b^2+b+1]})$$

$$E(X_{n+1} \mid X_n = y) = 1 + \frac{b^2 y+1}{b^2+b+1}. \qquad \text{(III.D.5.5)}$$

It is quite surprising that the regression of X_{n+1} on X_n is linear in y, considering the non-linearity of the process which generates the $\{X_n\}$.

The conditional expectation of X_n given X_{n+1} can be derived with equal dispatch.

Case	$E(X_n \mid X_{n+1} = y)$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{b+1}{b}$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n$	$\frac{y}{b}$
$X_n = [b+1]E_{n-1}, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$b+1$
$X_n = [b+1]E_{n-1}, X_{n+1} = [b+1]E_n$	$b+1$

Using III.D.5.1 through III.D.5.4 as before

$$E(X_n \mid X_{n+1} = y) = \sum_{i=1}^{4} E(X_n \mid X_{n+1} = y, \text{case } i) P(\text{case } i)$$

$$= (\frac{b+1}{b})(\frac{b^3}{[b+1][b^2+b+1]}) + (\frac{y}{b})(\frac{b}{b^2+b+1})$$

$$+ (b+1)(\frac{b}{b^2+b+1}) + (b+1)(\frac{1}{[b+1][b^2+b+1]})$$

$$E(X_n \mid X_{n+1} = y) = \frac{b^2+y}{b^2+b+1} + 1. \qquad \text{(III.D.5.6)}$$

The probability that X_{n+1} is greater than X_n can also be easily computed if one is careful to differentiate between the case where $b \leq 1$ and $b > 1$.

Case	$P(X_{n+1} > X_n)$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{1}{2}$

$$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n \qquad \begin{cases} 0 & \text{if } b \leq 1, \\ 1 & \text{if } b > 1 \end{cases}$$

$$X_n = [b+1]E_{n-1}, X_{n+1} = [\frac{b+1}{b}]E_{n+1} \qquad \frac{1}{b+1}$$

$$X_n = [b+1]E_{n-1}, X_{n+1} = [b+1]E_n \qquad \frac{1}{2}$$

Thus when $b \leq 1$ we have, using III.D.5.1 through III.D.5.4

$$P(X_{n+1} > X_n) = \sum_{i=1}^{4} P(X_{n+1} > X_n | \text{case } i) P(\text{case } i)$$

$$= (\frac{1}{2})(\frac{b^3}{[b+1][b^2+b+1]}) + (\frac{1}{b+1})(\frac{b}{b^2+b+1})$$

$$+ (\frac{1}{2})(\frac{1}{[b+1][b^2+b+1]})$$

$$P(X_{n+1} > X_n) = \frac{1}{2} - \frac{b^2}{(b+1)(b^2+b+1)}, \quad b \leq 1. \qquad \text{(III.D.5.7)}$$

A similar computation with $b > 1$ again using III.D.5.1 through III.D.5.4 yields

$$P(X_{n+1} > X_n) = \frac{1}{2} + \frac{b}{(b+1)(b^2+b+1)}, \quad b > 1. \qquad \text{(III.D.5.8)}$$

Thus a graph of $P(X_{n+1} > X_n)$ will have a discontinuity at $b = 1$ when case 2 switches from a probability of zero to a probability of one. This graph is presented as Figure III.D.5.1. The

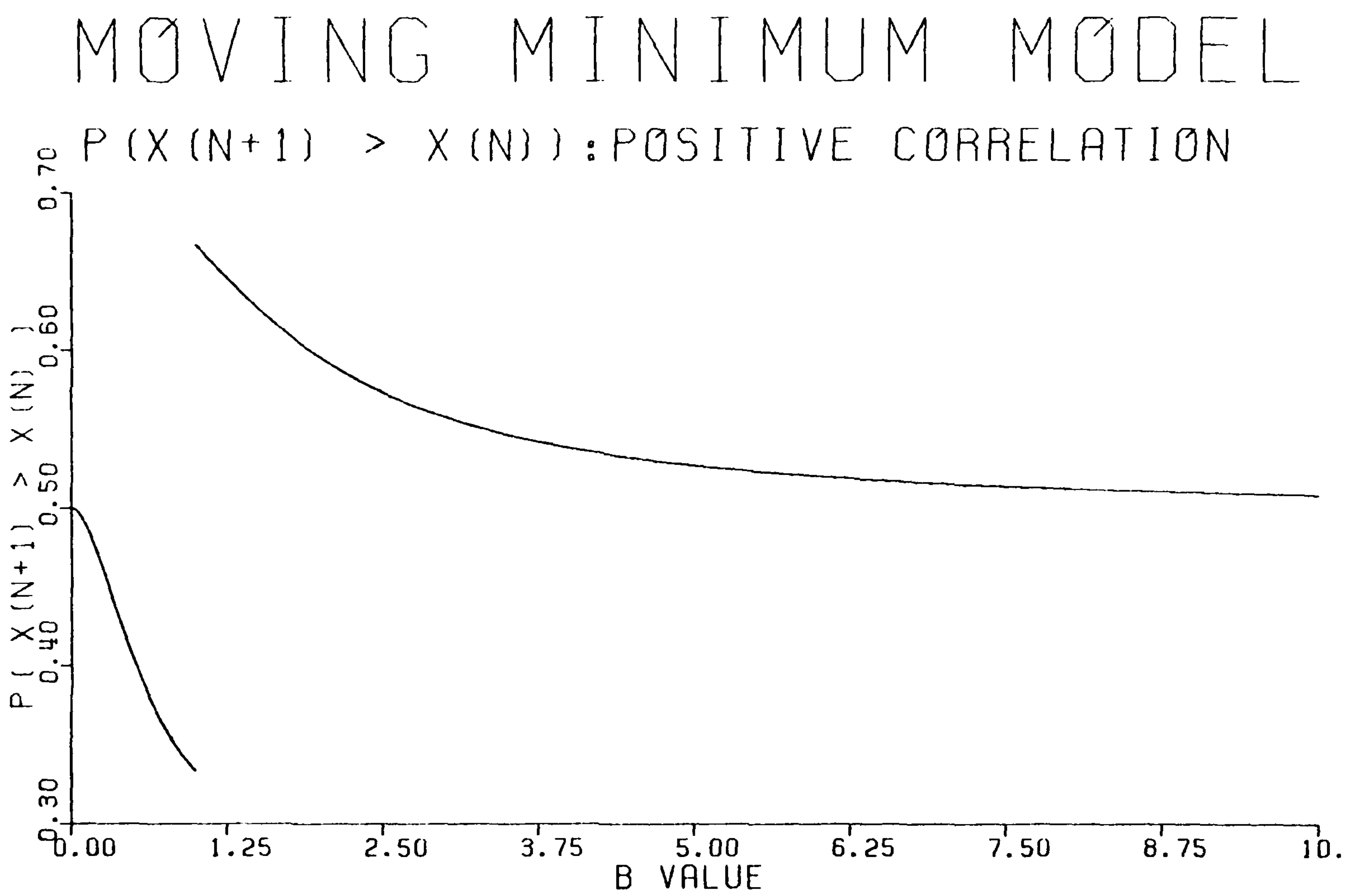


FIGURE III.D.5.1

minimum value of one-third occurs at $b = 1$. The maximum value of two-thirds occurs at $b = 1^+$. The moving minimum model, therefore, has a greater range of values for the $P(X_{n+1} > X_n)$ than does the NEMA(1) model. However, the greater range for the $P(X_{n+1} > X_n)$ and greater range of correlations must be balanced against the degeneracy of the model.

As was noted with the NEMA(1) model, the correlation in non-normal models does not define the joint properties of X_n and X_{n+1}. Although the cases of b and $\frac{1}{b}$ are indistinguishable from the viewpoint of correlation (see III.D.2.4 and III.D.3.5), these cases will have significantly different sample paths as indicated by III.D.5.7, III.D.5.8, and the discussion of runs up and down in III.D.4.

Three examples of sample paths for different b values are given in Figures III.D.5.2 through III.D.5.4. The degeneracy of the model is clearly present in the sample paths as a tendency to produce runs of equal, increasing or decreasing values, respectively. A comparison of Figures III.D.5.3 and III.D.5.4 quickly demonstrates that while these two sample paths have the same correlation, they produce significantly different $\{X_n\}$ sequences. This is a graphic indication that non-normal processes are not determined solely by their correlation structure.

Figures III.D.5.5 through III.D.5.7 are the scatter plots associated with the sample paths in Figures III.D.5.2 through III.D.5.4, respectively. Here, too, the degeneracy

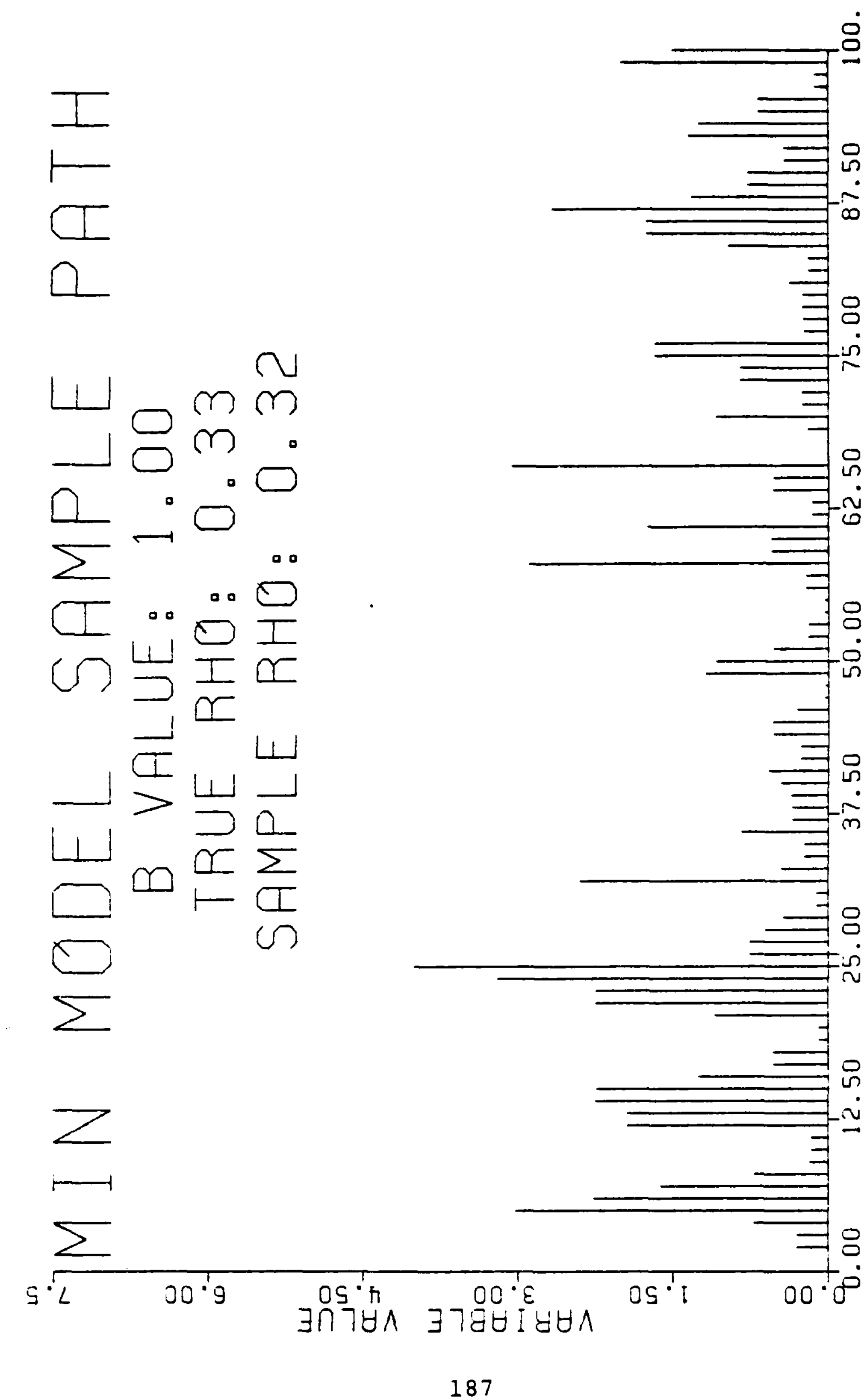


FIGURE III.D.5.2

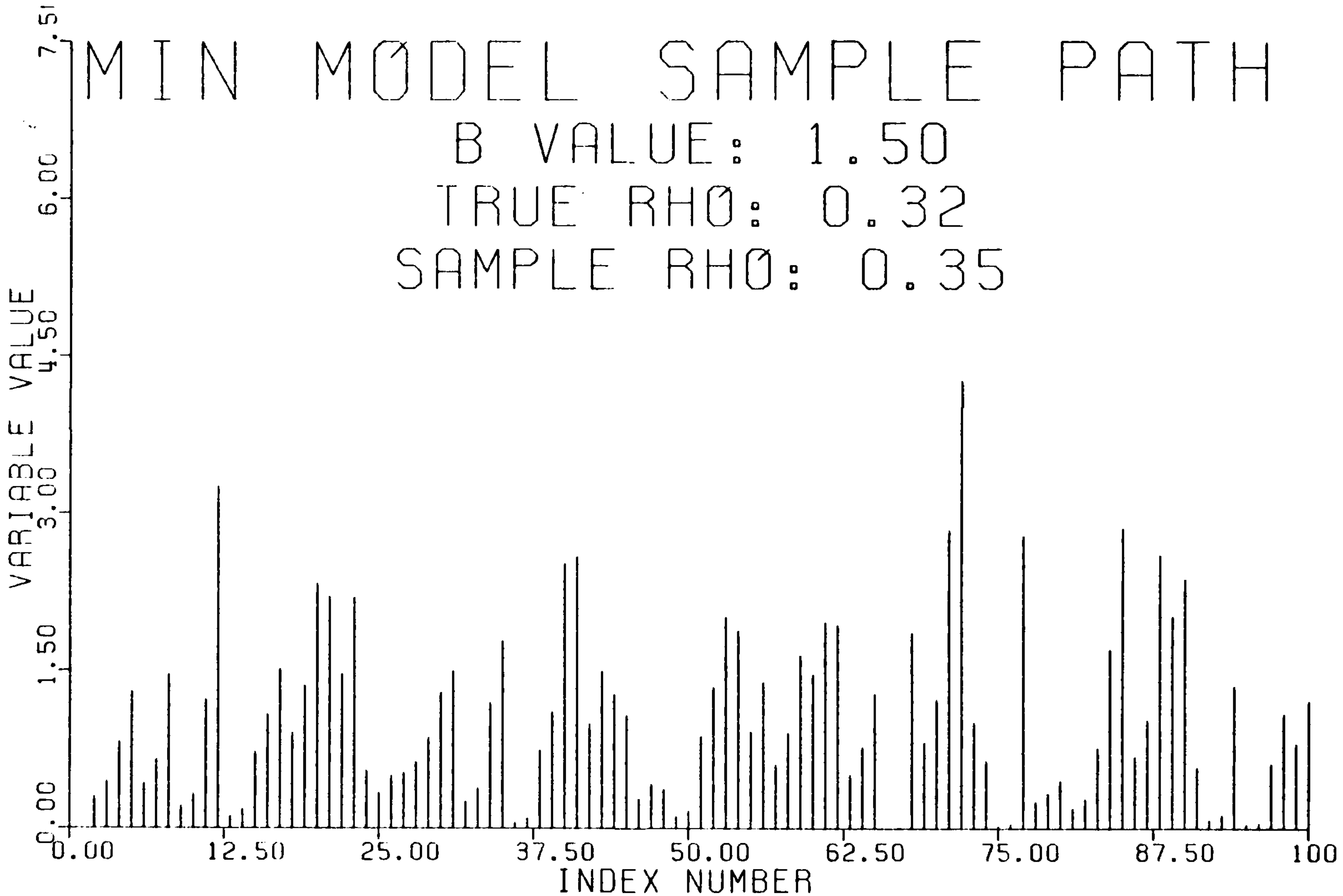


FIGURE III.D.5.3

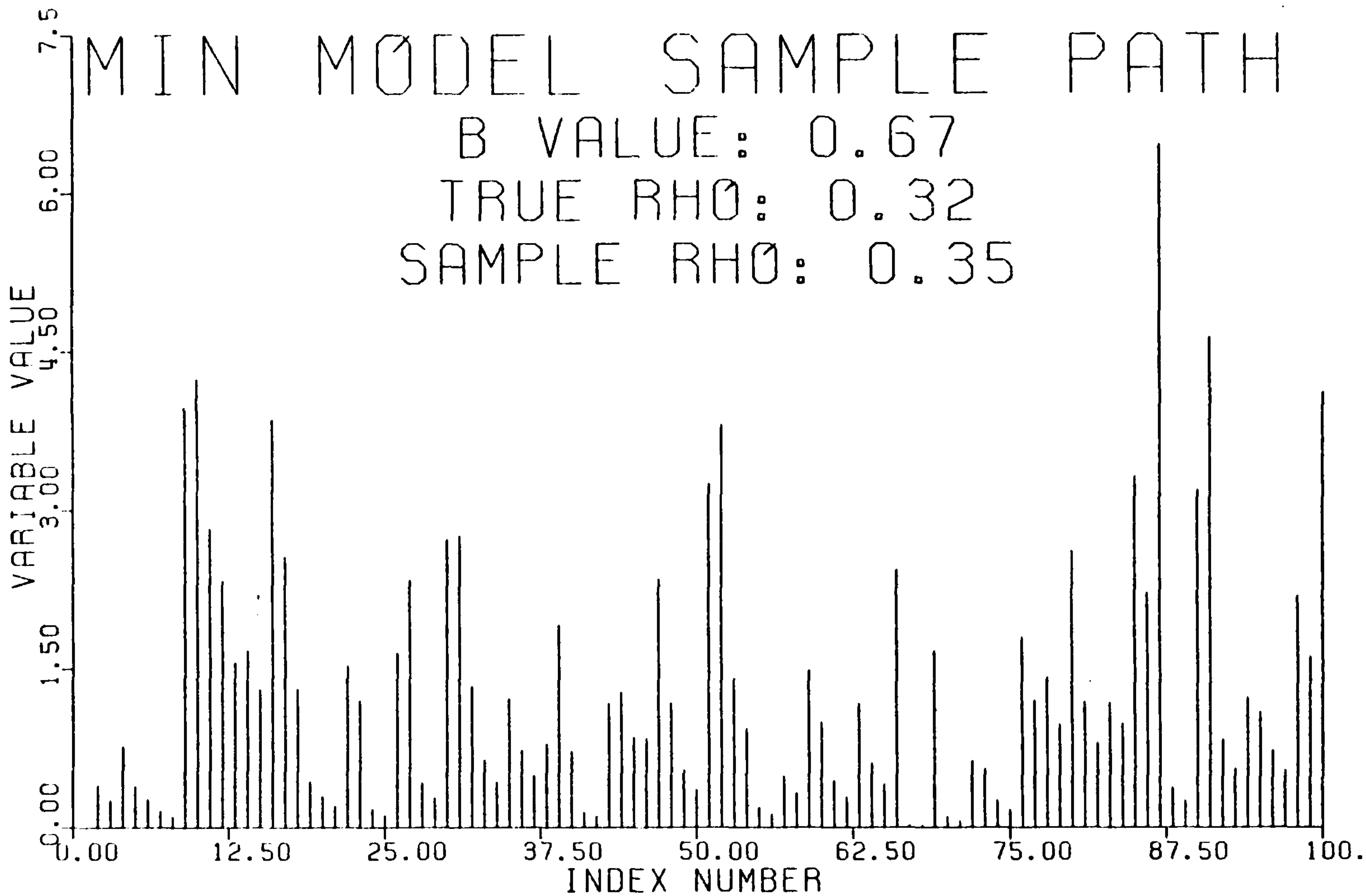


FIGURE III.D.5.4

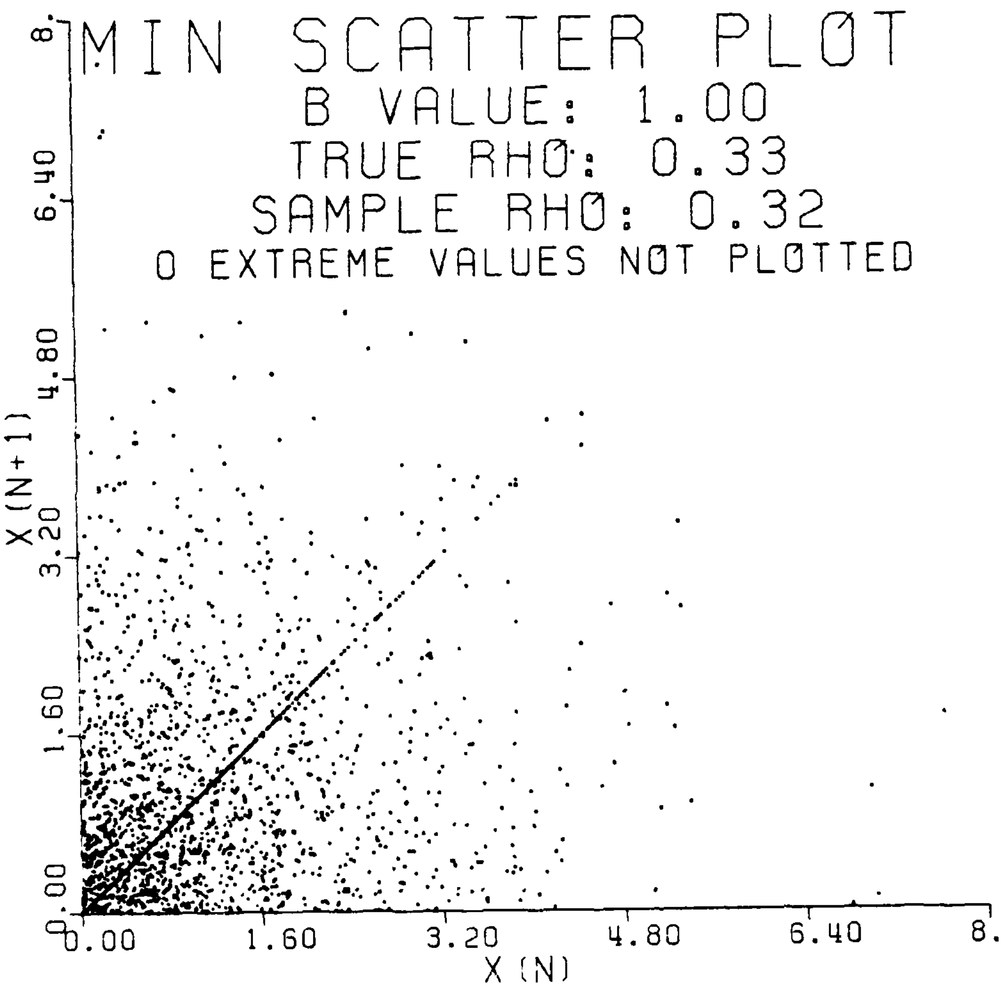


FIGURE III.D.5.5

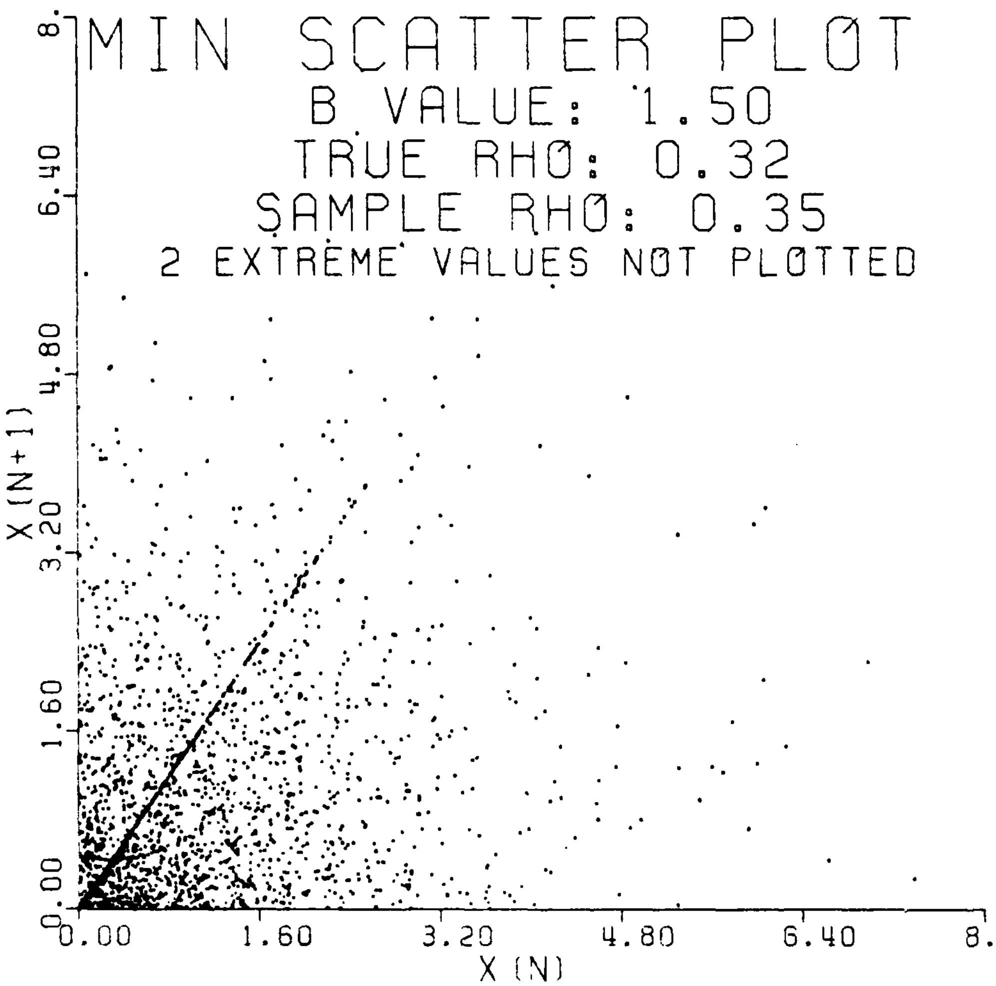
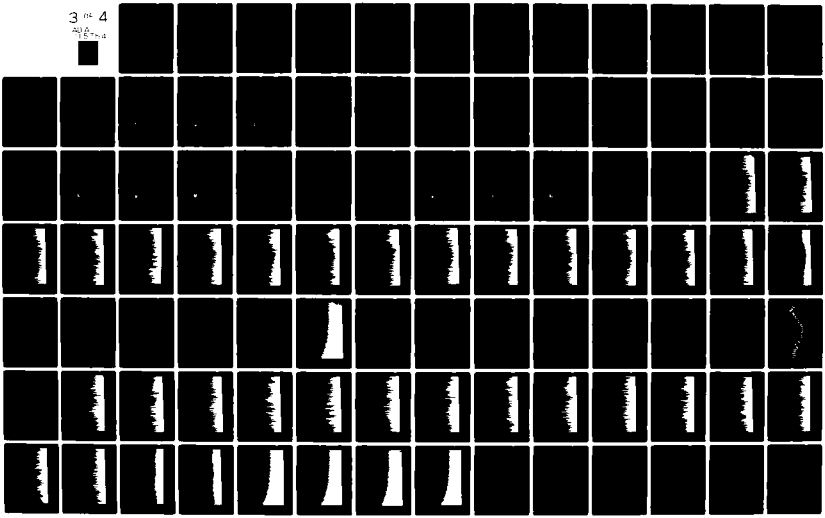


FIGURE III.D.5.6

AD-A115 754 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 12/1
EXTENSION OF SOME MODELS FOR POSITIVE-VALUED TIME SERIES.(U)
MAR 82 D K HUGUS

UNCLASSIFIED NL

3 OF 4
AD A
115754



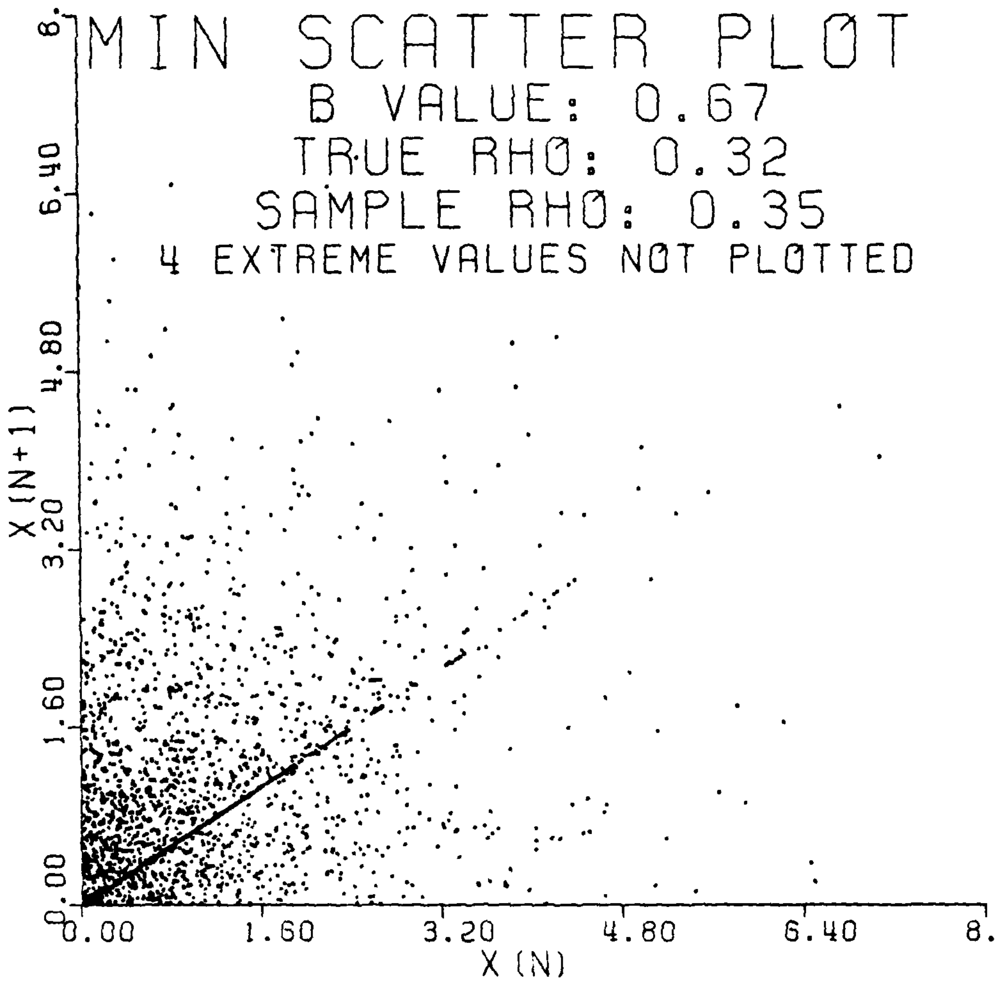


FIGURE III.D.5.7

of the model is clearly present in the tendency of the (X_n, X_{n+1}) plots to lie on the line $X_{n+1} = bX_n$. The slope of this line determines whether the runs are of equal, increasing or decreasing value.

6. Conditional Expectation and $P(X_{n+1} > X_n)$ for Antithetic Variables

Results similar to those obtained in III.D.5 can be obtained for the moving minimum model with negative correlation. The procedure for determining the conditional expectations and the probability that X_{n+1} is greater than X_n using antithetic variables is exactly the same as that in the previous section. First, the probability of each of the four possible combinations of X_n and X_{n+1} values is computed, the conditional expectation or probability is computed for each case, and the final weighted sum of conditional expectations or probabilities is finally computed. In one instance no closed form answer is available and numerical procedures are used.

Recall that the generation scheme when using antithetic variables is

$$X_n = \mathrm{MIN}\left(\left[\frac{b+1}{b}\right]E_n, [b+1]E'_{n-1}\right), \qquad \text{(III.D.6.1)}$$

where $\{E_n, n = 0,1,\ldots\}$ is an iid sequence of Exponentially distributed random variables with unit mean, $\{E'_n, n = 0,1,\ldots\}$ is generated from the $\{E_n\}$ sequence by the relationship $E'_n = -\ln(1-e^{-E_n})$ which implies that $\{E_n\}$ is also iid Exponential with unit mean, $b \geq 0$.

First, consider the case where $X_n = [\frac{b+1}{b}]E_n$ and $X_{n+1} = [\frac{b+1}{b}]E_{n+1}$. Then

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= P([\tfrac{b+1}{b}]E_n \le [b+1]E'_{n-1}, [\tfrac{b+1}{b}]E_{n+1} \le [b+1]E'_n)$$

Using the standard conditioning argument produces

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \int_0^\infty P([\tfrac{b+1}{b}]y \le [b+1]E'_n, [\tfrac{b+1}{b}]E_{n+1} \le -[b+1]\ln(1-e^{-y}) \mid E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E'_{n-1} > \tfrac{y}{b})P(E_{n+1} \le -b\ln[1-e^{-y}])e^{-y}dy$$

$$= \int_0^\infty e^{-y/b}(1-[1-e^{-y}]^b)e^{-y}dy$$

$$= \int_0^\infty e^{-(1+\frac{1}{b})y}dy - \int_0^\infty (1-e^{-y})^b(e^{-y})^{(1+\frac{1}{b})}dy$$

The first integral is straightforward. In the second integral, the change of variable $u = e^{-y}$, $-\frac{du}{u} = dy$ makes this integral recognizable as the integral of a Beta random variable. Making this change of variable and making the appropriate changes in the limits of integration produces

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [b+1]E_{n+1})$$

$$= \frac{b}{b+1} - \int_0^1 (1-u)^b (u)^{\frac{1}{b}} du$$

$$= \frac{b}{b+1} - \frac{\Gamma(1+b)\Gamma(1+\frac{1}{b})}{\Gamma(2+b+\frac{1}{b})}$$

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [b+1]E_{n+1})$$

$$= \frac{b}{b+1} - \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} \qquad \text{(III.D.6.2)}$$

In the second case, $X_n = [\frac{b+1}{b}]E_n$ and $X_{n+1} = [b+1]E_n'$. Proceeding as before

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n')$$

$$= P([\tfrac{b+1}{b}]E_n \leq [b+1]E_{n-1}', [b+1]E_n' \leq [\tfrac{b+1}{b}]E_{n+1})$$

$$= \int_0^\infty P([\tfrac{b+1}{b}]y \leq [b+1]E_{n-1}', -[b+1]\ln[1-e^{-y}] \leq [\tfrac{b+1}{b}]E_{n+1} | E_n = y)\, e^{-y} dy$$

$$= \int_0^\infty P(E_{n-1}' > \tfrac{y}{b})\, P(E_{n+1} > -b\ln[1-e^{-y}])\, e^{-y} dy$$

$$= \int_0^\infty e^{-y/b}(1-e^{-y})^b e^{-y} dy$$

$$= \int_0^\infty (1-e^{-y})^b e^{-(1+\frac{1}{b})y} dy$$

This is the same integral as the second integral in the first case. Thus

$$P(X_n = [\tfrac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n')$$

$$= \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} \qquad \text{(III.D.6.3)}$$

Next consider the case where $X_n = (b+1)E_{n-1}'$ and $X_{n+1} = (\frac{b+1}{b})E_{n+1}$. Then

$$P(X_n = [b+1]E_{n-1}', X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= P([b+1]E_n' \leq [\tfrac{b+1}{b}]E_n, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]E_n'$$

$$= \int_0^\infty P([b+1]E_n' \leq [\tfrac{b+1}{b}]y, [\tfrac{b+1}{b}]E_{n+1} \leq [b+1]\ln[1-e^{-y}] \mid E_n = y)e^{-y}dy$$

$$= \int_0^\infty P(E_n' \leq \tfrac{y}{b})P(E_{n+1} \leq -b\ln[1-e^{-y}])e^{-y}dy$$

$$= \int_0^\infty e^{-y}dy - \int_0^\infty e^{-(1+\frac{1}{b})y}ey - \int_0^\infty e^{-y}(-e^{-y})^b dy + \int_0^\infty (1-e^{-y})^b(e^{-y})^{(1+\frac{1}{b})}dy$$

The first two integrals do not present a problem. Making the change of variable $z = 1-e^{-y}$, $dz = e^{-y}dy$ makes the third integral easy. The last integral is the same as the second integral in the first case. So

$$P(X_n = [b+1]E'_{n-1}, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \quad 1 - \frac{b}{b+1} - \frac{1}{b+1} + \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})}$$

$$P(X_n = [b+1]E'_{n-1}, X_{n+1} = [\tfrac{b+1}{b}]E_{n+1})$$

$$= \quad \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} \tag{III.D.6.4}$$

Finally, consider the case when $X_n = (b+1)E'_{n-1}$ and $X_{n+1} = (b+1)E'_n$. Then

$$P(X_n = [b+1]E'_{n-1}, X_{n+1} = [b+1]E'_n)$$

$$= \quad P([b+1]E'_n \leq [\tfrac{b+1}{b}]E_n, [b+1]E'_n \leq [\tfrac{b+1}{b}]E_{n+1})$$

$$= \quad \int_0^\infty P([b+1]E'_n \leq [\tfrac{b+1}{b}]y, -[b+1]\ln[1-e^{-y}] \leq [\tfrac{b+1}{b}]E_{n+1} \mid E_n = y)\, e^{-y}dy$$

$$= \quad \int_0^\infty P(E'_{n-1} \leq \tfrac{y}{b})\, P(E_{n+1} \quad -b\ln[1-e^{-y}])\, e^{-y}dy$$

$$= \quad \int_0^\infty (1-e^{-y})^b e^{-y}dy - \int_0^\infty (1-e^{-y})^b (e^{-y})^{(1+\frac{1}{b})}\, dy$$

These integrals are the same as the third and fourth integrals in case three. Therefore,

$$P(X_n = [b+1]E'_{n-1}, X_{n+1} = [b+1]E'_n)$$

$$= \frac{1}{b+1} - \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})} \qquad \text{(III.D.6.5)}$$

The conditional expectations given a specific case of the values of X_n and X_{n+1} can be written by inspection. Hence,

Case	$E(X_{n+1} \mid X_n = y)$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1};$	$\frac{b+1}{b}.$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [b+1]E'_n;$	$-(b+1)\ln(1-e^{-\frac{by}{b+1}}).$
$X_n = [b+1]E'_{n-1}, X_{n+1} = [\frac{b+1}{b}]E_{n+1};$	$\frac{b+1}{b}.$
$X_n = [b+1]E'_{n-1}, X_{n+1} = [b+1]E'_n;$	$b+1.$

Combining these results with equations III.D.6.2 through III.D.6.5 and letting

$$G = \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})}$$

we have

$$E(X_{n+1}|X_n = y) = \sum_{i=1}^{4} E(X_{n+1}|X_n = y, \text{case } i)P(\text{case } i) \qquad \text{(III.D.6.6)}$$

$$= \left(\frac{b}{b+1} - G\right)\left(\frac{b+1}{b}\right) - G(b+1)\ln\left(1-e^{-\frac{by}{b+1}}\right)$$

$$+ G\left(\frac{b+1}{b}\right) + \left(\frac{1}{b+1} - G\right)(b+1)$$

$$E(X_{n+1}|X_n = y) = 2 - G(b+1)\left(1+\ln\left[1-e^{-\frac{by}{b+1}}\right]\right) \qquad \text{(III.D.6.7)}$$

Similarly, we can derive the expression for $E(x_n|X_{n+1} = y)$

<u>Case</u>	$\underline{E(X_n \mid X_{n+1} = y)}$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$;	$\frac{b+1}{b}$.
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [b+1]E_n'$;	$-(\frac{b+1}{b})\ln(1-e^{-\frac{y}{b+1}})$.
$X_n = [b+1]E_{n-1}', X_{n+1} = [\frac{b+1}{b}]E_{n+1}$;	$(b+1)$.
$X_n = [b+1]E_{n-1}', X_{n+1} = [b+1]E_n'$;	$(b+1)$.

Then using III.D.6.2 through III.D.6.5 and again letting

$$G = \frac{b^2\Gamma(b)\Gamma(\frac{1}{b})}{(b^2+b+1)(b^2+1)\Gamma(b+\frac{1}{b})}$$

$$E(X_n|X_{n+1} = y) = \sum_{i=1}^{4} E(X_n|X_{n+1}, \text{case } i)P(\text{case } i)$$

$$= \left(\frac{b+1}{b}\right)\left(\frac{b}{b+1}-G\right) - G\left(\frac{b+1}{b}\right)\ln\left(1-e^{-\frac{y}{b+1}}\right) + (b+1)G + (b+1)\left(\frac{1}{b+1}-G\right)$$

$$E(X_n | X_{n+1} = y) = 2 - G(\frac{b+1}{b})(1 + \ln[1-e^{-\frac{y}{b+1}}]) \qquad \text{(III.D.6.8)}$$

The probability that X_{n+1} is greater than X_n can be approached in the same fashion as the conditional expectations. The second case will be reserved for individual attention.

Case	$P(X_n+1 > X_n)$
$X_n = [\frac{b+1}{b}]E_n, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{1}{2}$
$X_n = [b+1]E'_{n-1}, X_{n+1} = [\frac{b+1}{b}]E_{n+1}$	$\frac{1}{b+1}$
$X_n = [b+1]E'_{n-1}, X_{n+1} = [b+1]E'_n$	$\frac{1}{2}$

The second case, $X_n = [\frac{b+1}{b}]E_n$ and $X_{n+1} = [b+1]E'_n$, does not allow a closed form solution. We get

$$\begin{aligned}
P(X_{n+1} > X_n) &= P([b+1]E'_n > [\frac{b+1}{b}]E_n) \\
&= P(-[b+1]\ln[1-e^{-E_n}] > [\frac{b+1}{b}]E_n) \\
&= P(-\ln[1-e^{-E_n}] > \frac{E_n}{b}) \\
&= P([1-e^{-E_n}]^b < e^{-E_n})
\end{aligned}$$

Hence, if we can find y_0 such that $(1-e^{-y_0})^b = e^{-y_0}$, then $P(X_{n+1} > X_n) = 1-e^{-y_0}$. The required solution can be found by numerical means for any given value of b. A computer program to determine y_0 to an accuracy of 10^{-6} for a given value of b and to compute $P(X_{n+1} > X_n)$ was prepared. A graph of the results is presented as Figure III.D.6.1. When using antithetic variables, the moving minimum model has a restricted range of possible values for $P(X_{n+1} > X_n)$. The maximum value of approximately 0.509 occurs at about 0.30. The minimum value of approximately 0.491 occurs at about 3.33.

This small range of values for the $P(X_{n+1} > X_n)$ is shown in the relative indistinguishability among the sample paths displayed in Figures III.D.6.2 through III.D.6.4. Of more interest are the scatter plots presented in Figures III.D.6.5 through III.D.6.7. In these plots the degeneracy of the moving minimum model is clearly displayed. When X_n achieves a value of y based on E_n, then X_{n+1} is constrained to have a value less than $-\ln(1-e^{-X_n})$. In the case where equality is achieved, the second case in the discussion in this section, the point (X_n, X_{n+1}) lies on the curve $e^{-X_n} + e^{-X_{n+1}} = 1$. This constraint is clearly evident in the scatter plots. Thus the moving minimum model displays a degenerate behavior for negative correlations as well as positive.

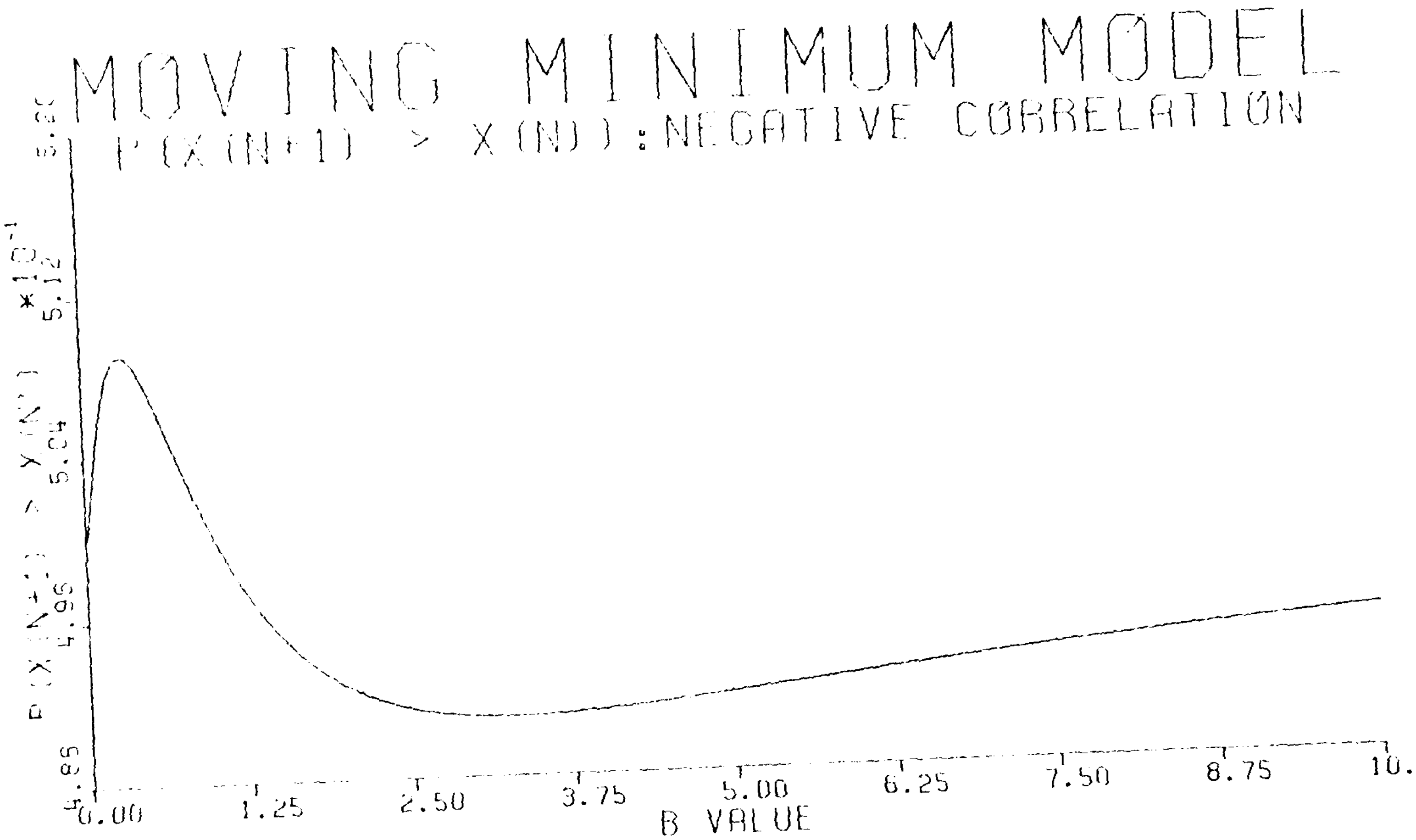


FIGURE III.D.6.1

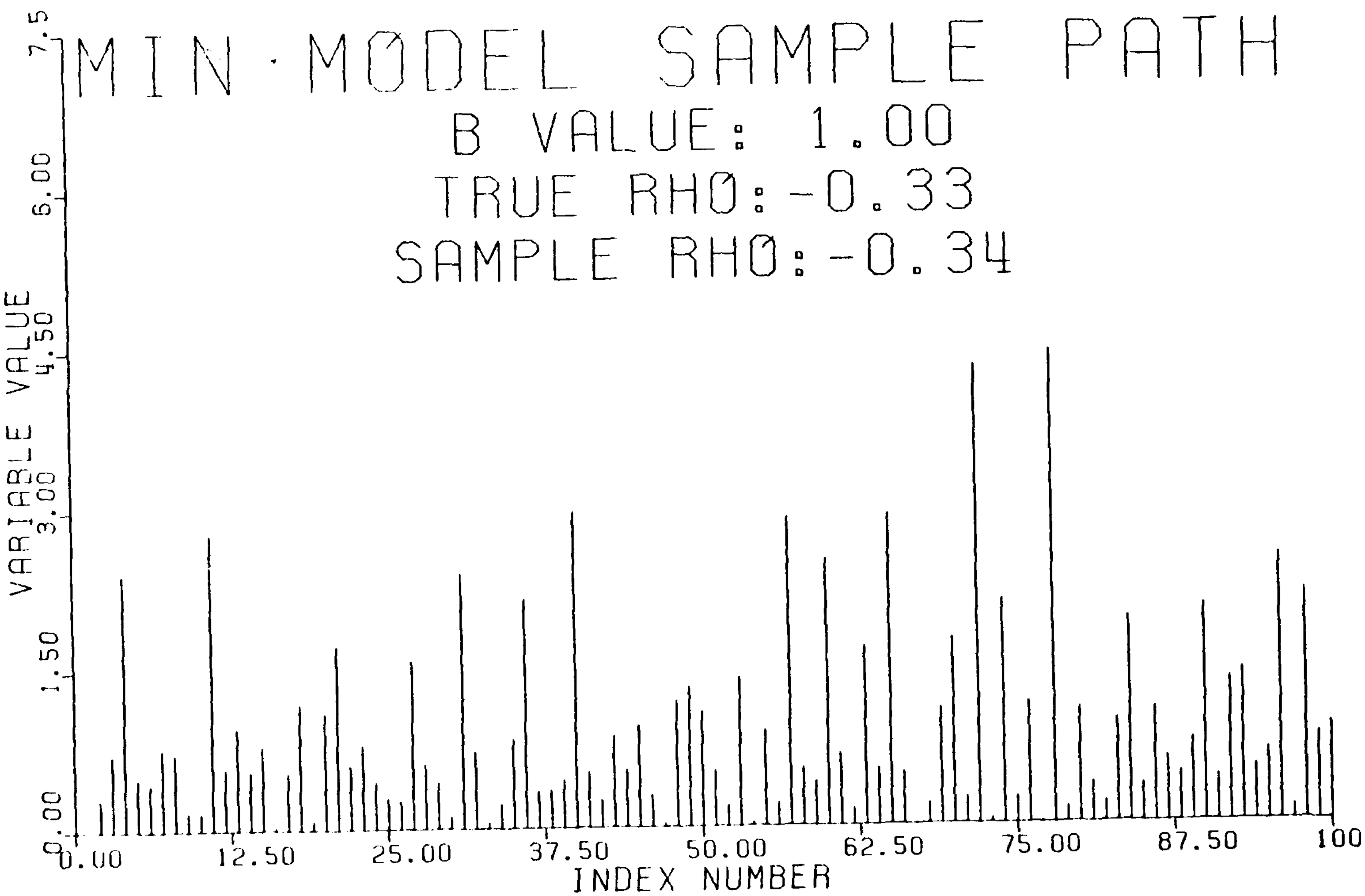


FIGURE III.D.6.2

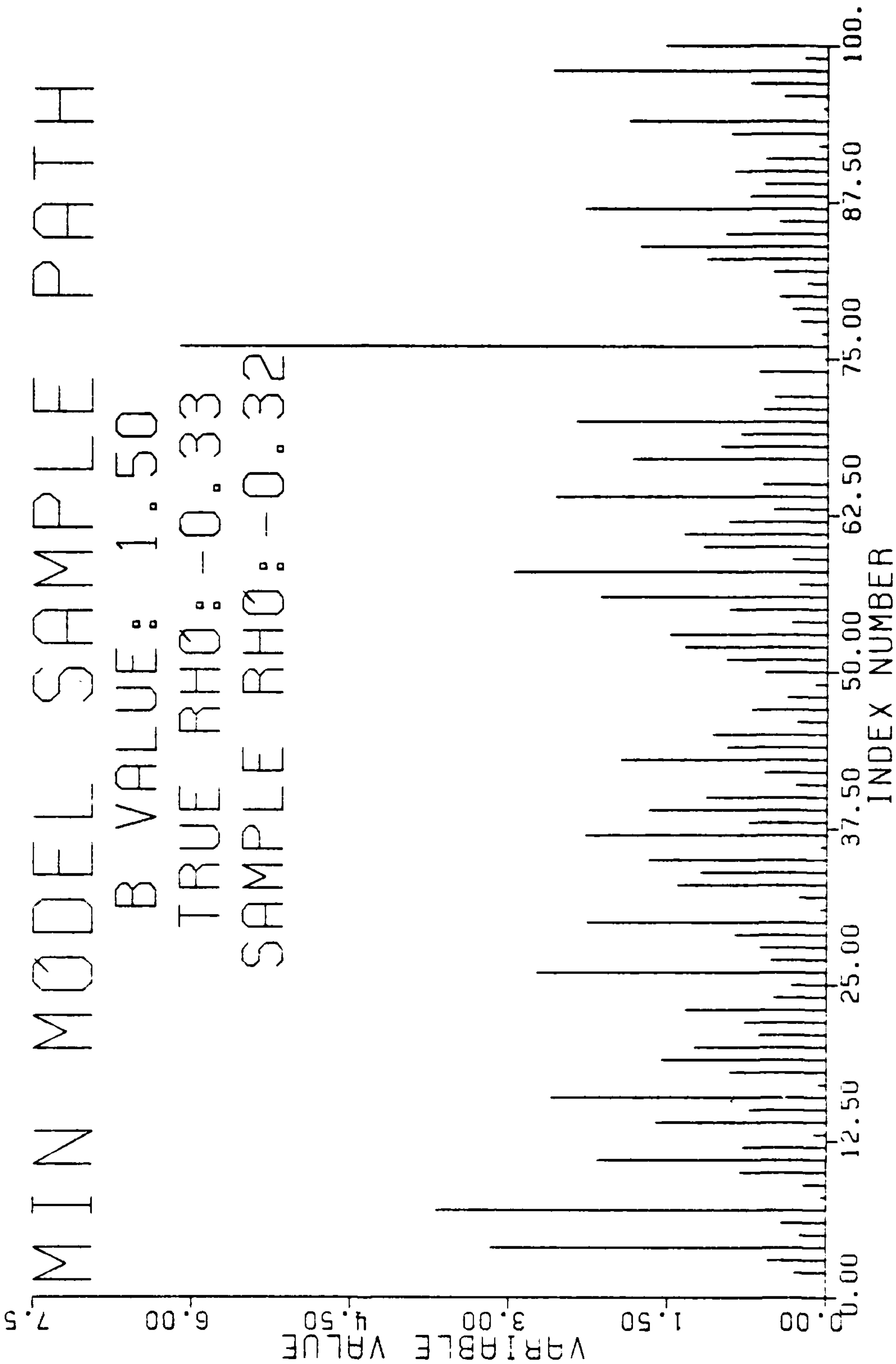


FIGURE III.D.6.3

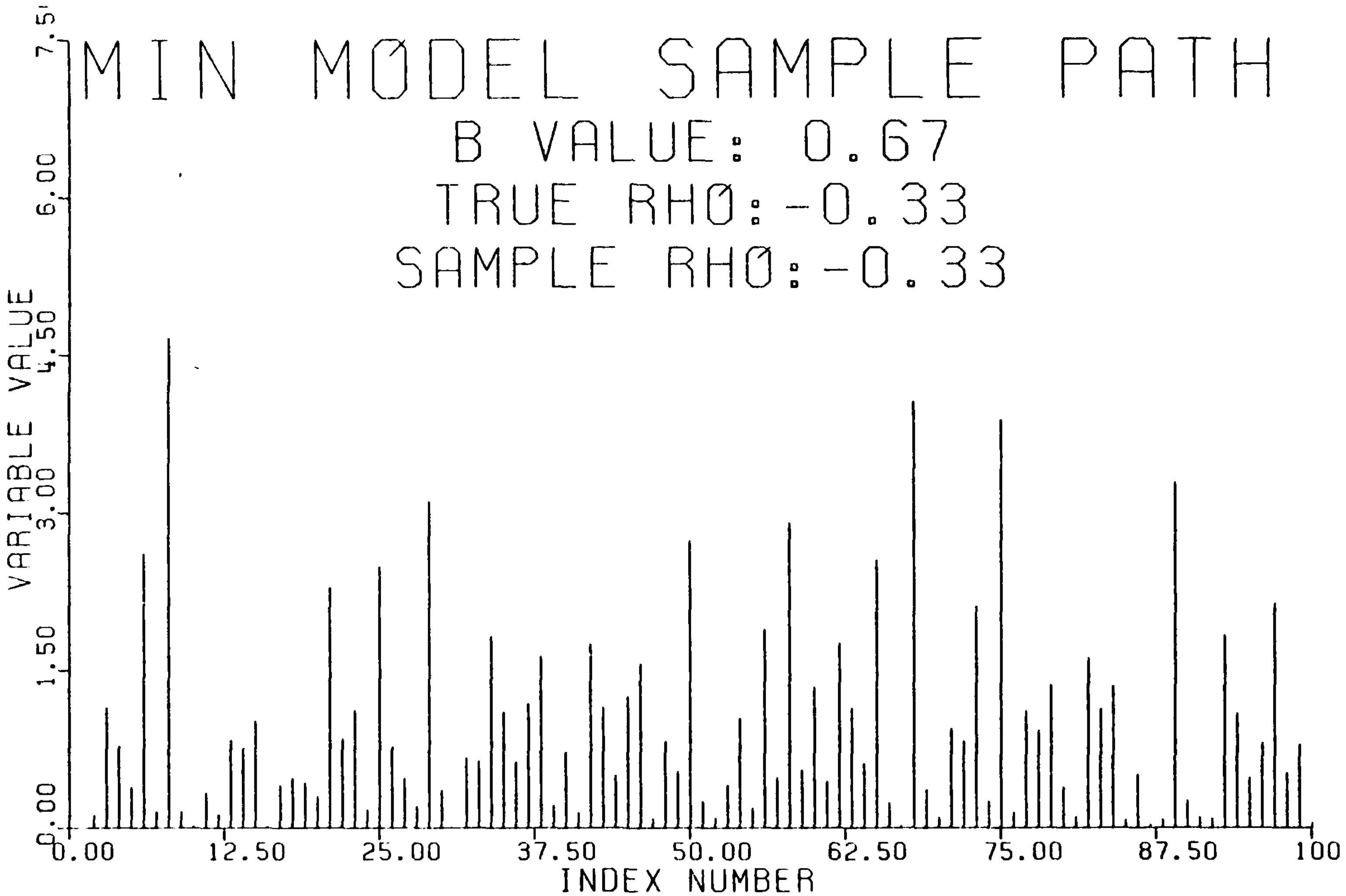


FIGURE III.D.6.4

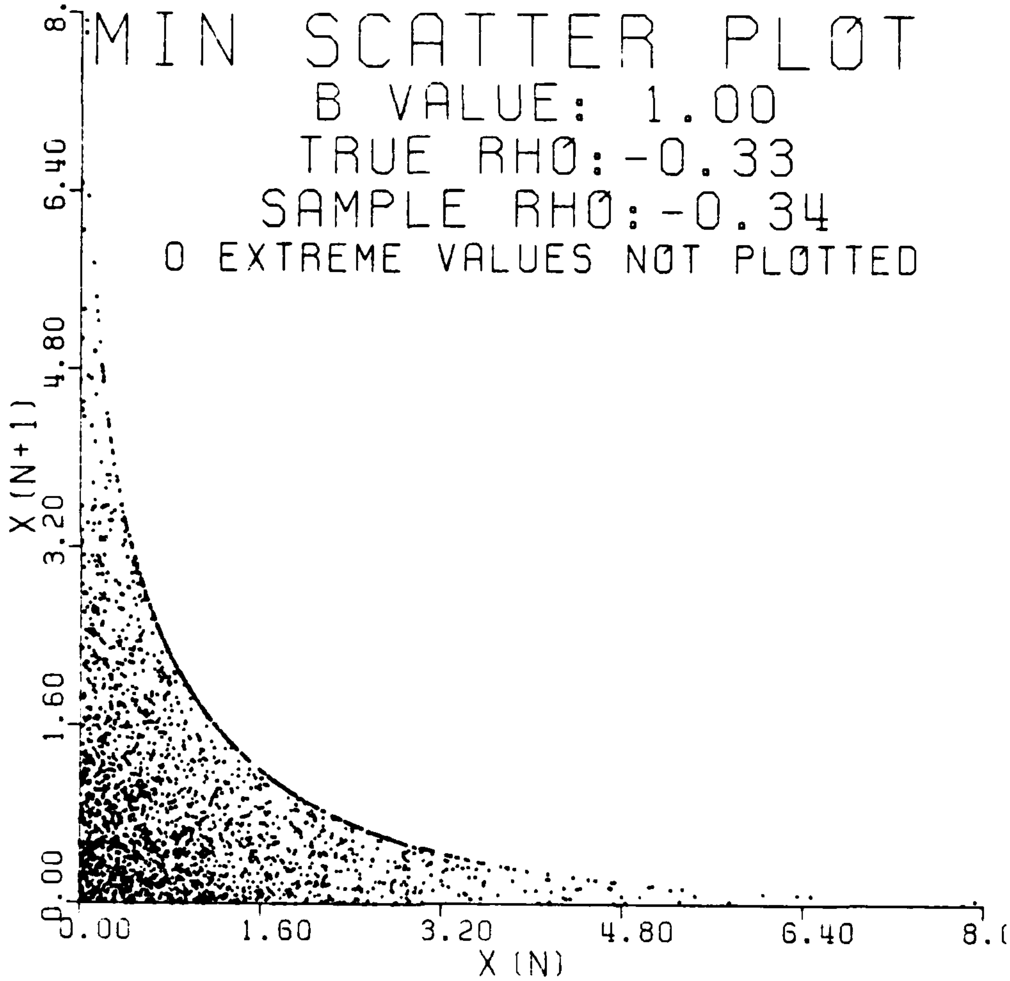


FIGURE III.D.6.5

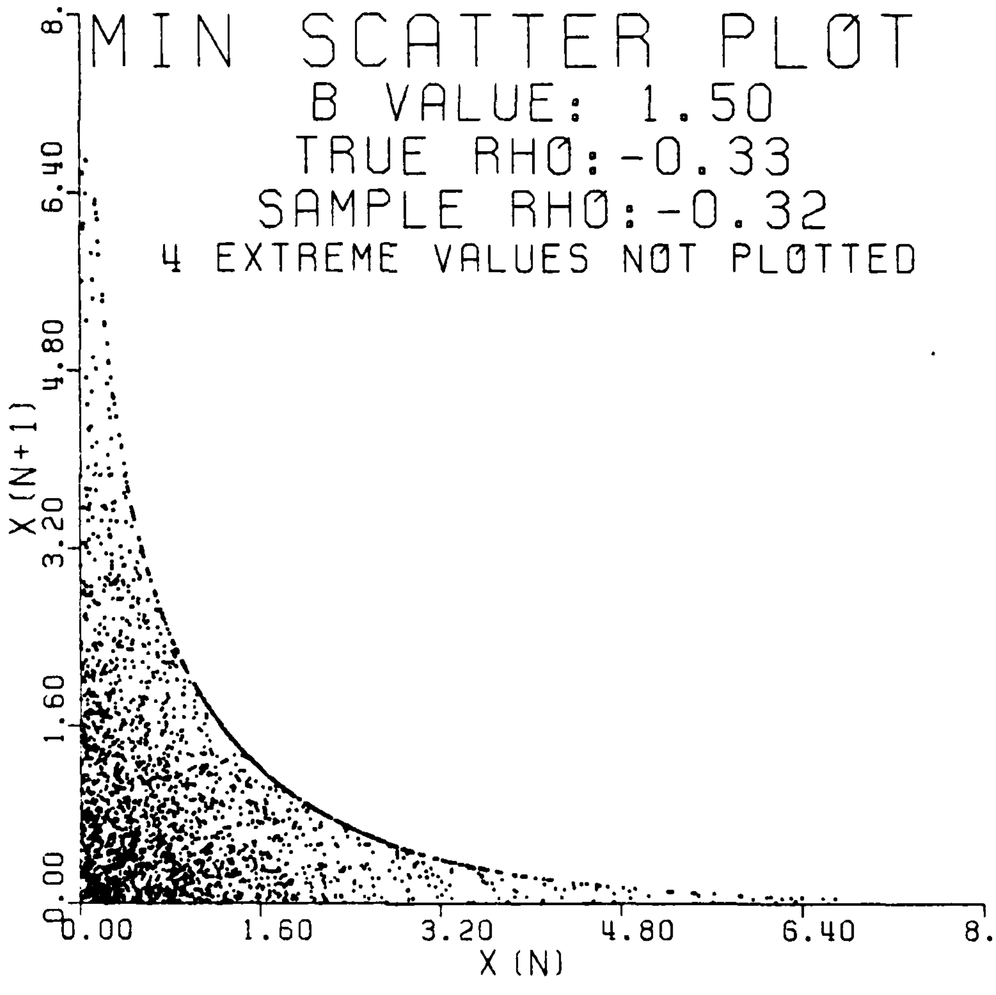


FIGURE III.D.6.6

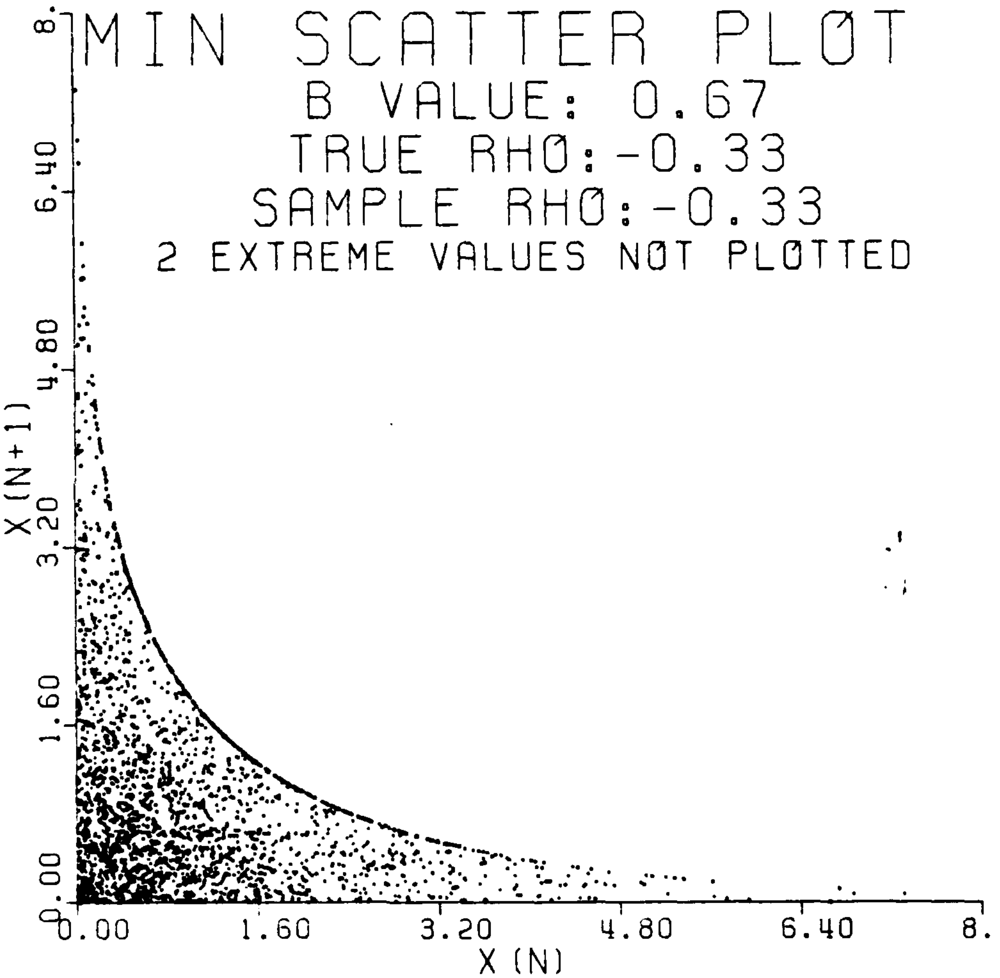


FIGURE III.D.6.7

E. THE BETA-EXPONENTIAL MODEL

1. Introduction

A third method that can be used to generate correlated, marginally Exponentially distributed random variables is a special case of the Beta-Gamma model given in Lawrance and Lewis [Ref. 18]. This model generates an $\{X_n\}$ sequence using the relation

$$X_n = B_n(q,1-q)E_n + B_n(1-q,q)E_{n-1} \qquad n = 1,2,\ldots, \qquad \text{(III.E.1.1)}$$

where $\{B_n(q,1-q),\ n = 1,2,\ldots\}$ is an iid sequence of Beta random variables, $\{B_n(1-q,q),\ n = 1,2,\ldots\}$ is an iid sequence of Beta random variables, $\{E_n,\ n = 0,1,\ldots\}$ is an iid sequence of Exponential random variables with unit mean, $\{B_n(q,1-q)\}$, $\{B_n(1-q,q)\}$, and $\{E_n\}$ are mutually independent, and $0 < q < 1$. The density for a Beta(m,n) variable is

$$\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1}(1-x)^{n-1} \quad 0 \le x \le 1;\ m > 0;\ n > 0. \qquad \text{(III.E.1.2)}$$

In practice the Beta random variables can be generated from two Gamma distributed random variables using the relationship $B(m,n) = \frac{G(m)}{G(m)+G(n)}$ from Kotz and Johnson [Ref. 19], where G(K) is a Gamma random variable with a shape parameter of K and a scale parameter of one.

This is a special case of the Gamma model considered in Chapter II of this thesis. It works because, as described by Lewis [Ref. 10], in III.E.1.1 the product of the $B_n(q,1-q)$

variable and the E_n variable is a Gamma (q) variable. Similarly, the product of the $B_n(1-q,q)$ and the E_{n-1} variables is a Gamma (1-q), independent of the Gamma (q) variable. Consequently, their sum is an Exponential variable, X_n. The $\{X_n\}$ process is clearly one-dependent, as for the NEMA(1) process.

Because of a lack of a closed form solution for the integral of the Beta density when the limits of integration are not from zero to one, this model is the least tractable of those considered in Chapter III of this thesis. However, its correlation structure can be determined, an expression for the Laplace-Stieltjes transform of a sum of r random variables can be derived, and the probability of X_{n+1} being greater than X_n can be simulated. An advantage of this model is that it extends directly to moving average Gamma processes (see Chapter II). This extension is not possible with the NEMA(1) or the moving minimum model.

2. Correlation Structure, Positive and Negative

The correlation structure can be determined using a standard approach. We have using III.E.1.1

$$
\begin{aligned}
X_n X_{n+1} &= (B_n(q,1-q)E_n + B_n(1-q,q)E_{n-1})(B_{n+1}(q,1-q)E_{n+1} \\
&\quad + B_{n+1}(1-q,q)E_n) \\
&= B_{n+1}(q,1-q)B_n(q,1-q)E_{n+1}E_n + B_{n+1}(q,1-q)B_n(1-q,q)E_{n+1}E_{n-1} \\
&\quad + B_{n+1}(1-q,q)B_n(q,1-q)E_n^2 + B_{n+1}(1-q,q)B_n(1-q,q)E_nE_{n-1}
\end{aligned}
$$

Taking expectations and using the iid nature and independence of the $\{B_n(q,1-q)\}$, $\{B_n(1-q,q)\}$, and $\{E_n\}$ sequences yields

$$E(X_n X_{n+1}) = q^2 + q(1-q) + 2q(1-q) + (1-q)^2$$

Hence,

$$COV(X_n, X_{n+1}) = q(1-q)$$

and

$$CORR(X_n, X_{n+1}) = q(1-q), \quad 0 < q < 1. \qquad \text{(III.E.2.1)}$$

As with the other linear additive models, this correlation is double valued and lies in the range $(0,\frac{1}{4})$.

The range of possible correlations can be extended to negative values by modifying the generation formula by including an $\{E_n'\}$ sequence. Thus

$$X_n = B_n(q,1-q)E_n + B_n(1-q,q)E_{n-1}', \qquad \text{(III.E.2.2)}$$

where all random variables and constants are as defined for III.E.1.1 and $\{E_n', n = 0,1,\ldots\}$ is an iid sequence having a specified correlation with the $\{E_n\}$ sequence. In particular, E_n and E_n' may be an antithetic pair. The correlation of the $\{X_n\}$ using II.E.2.2 can be determined in the same way as before. Consequently,

$$X_n X_{n+1} = (B_n(q,1-q)E_n + B_n(1-q,q)E'_{n-1})$$

$$(B_{n+1}(q,1-q)E_{n+1} + B_{n+1}(1-q,q)E'_n)$$

$$= B_{n+1}(q,1-q)B_n(q,1-q)E_{n+1}E_n + B_{n+1}(q,1-q)B_n(1-q,q)E_{n+1}E'_{n-1}$$

$$+ B_{n+1}(1-q,q)B_n(q,1-q)E_nE'_n + B_{n+1}(1-q,q)B_n(1-q,q)E'_nE'_{n-1}$$

Taking expectations as before

$$E(X_nX_{n+1}) = q^2+q(1-q)+q(1-q)[COV(E_n,E'_n)+1]+(1-q)^2$$

$$= 1 + q(1-q)COV(E_n,E'_n)$$

Therefore,

$$COV(X_n,X_{n+1}) = q(1-q)COV(E_n,E'_n)$$

and

$$CORR(X_n,X_{n+1}) = q(1-q)CORR(E_n,E'_n), \quad 0 < q < 1. \qquad (III.E.2.3)$$

When $E'_n = E_n$, the correlation is one and III.E.2.3 reduces to III.E.2.2. When E_n and E'_n are antithetic the correlation is -0.6449 and negative correlations result. When q is 0.50, the correlation for the $\{X_n\}$ sequence falls in the (-0.16,0.25) range depending on the correlation between E_n and E'_n.

3. Laplace-Stieltjes Transform of a Sum

$$T_r = \sum_{i=1}^{r} X_i, \qquad \text{(III.3.1)}$$

where $\{X_i\}$ are defined by III.E.1.1. Then

$$T_r = X_1 + X_2 + \ldots + X_r$$

$$= B_1(q,1-q)E_1 + B_1(1-q,q)E_0 + B_2(q,1-q)E_2 + B_2(1-q,q)E_1$$

$$+ \ldots + B_r(q,1-q)E_r + B_r(1-q,q)E_{r-1}$$

$$= B_r(q,1-q)E_r + B_1(1-q,q)E_0 + [B_1(q,1-q) + B_2(1-q,q)]E_1$$

$$+ \ldots [B_{r-1}(q,1-q) + B_r(1-q,q)]E_{r-1}$$

Let $\phi_{T_r} = E(e^{-sT_r})$. Then using the iid nature and independence of $\{B_n(q,1-q)\}$, $\{B_r(1-q,q\}$, and $\{E_n\}$

$$\phi_{T_r}(s) = E(e^{-s[B_r(q,1-q)E_r + B_1(1-q,q)E_0 + \{B_1(q,1-q)+B_2(1-q,q)\}E_1 + \ldots + \{B_{r-1}(q,1-q)+B_r(1-q,q)\}E_{r-1}]})$$

$$= E(e^{-sB_r(q,1-q)E_n})E(e^{-sB_1(1-q,q)E_0})$$

$$\times E(e^{-s[(B_1(q,1-q)+B_2(1-q,q]E_1})$$

$$\times \ldots E(e^{-s[B_{r-1}(q,1-q)+B_r(1-q,q)]E_{r-1}})$$

$$\phi_{T_r}(s) = [\frac{1}{1+s}]^q [\frac{1}{1+s}]^{1-q} [\Psi_{BE}(s)]^{r-1}$$

$$= [\frac{1}{1+s}] [\Psi_{BE}(s)]^{r-1},$$

where

$$\Psi_{BE}(s) = E(e^{-s[B_i(q,1-q)+B_{i-1}(1-q,q)]E_{i-1}}).$$

The Laplace transform of the sum of two Beta random variables is a confluent hypergeometric function. Its form is too complicated to be of significant value in deriving the analytic behavior of the Beta-Exponential model.

4. <u>Empirical $P(X_{n+1} > X_n)$</u>

Because of the presence of the Beta random variables, the probability of X_{n+1} being greater can not be analytically determined with a reasonable amount of effort. In an attempt to establish a range for this probability, a simulation was used. In order to achieve a precision of at least 0.001, sixty-eight thousand comparisons were generated for each of ten random number seeds. The Beta random variables were generated using the Kotz and Johnson [Ref. 32] relation $B(m,n) = \frac{G(m)}{G(m)+G(n)}$ explained in III.E.1. The Exponential sequences were generated by a call to a standard generator of Exponentials. When the simulation was run for nineteen values of q from 0.05 to 0.95 in steps of 0.05, the $P(X_{n+1} > X_n)$ was 0.500 for all values of q.

Although the empirical probability that X_{n+1} is greater than X_n is constant at a value of 0.500, reminiscent of the autoregressive model of Chapter II.B.6, the distribution of $X_{n+1} - X_n$ is not symmetric and no simple proof for this situation has been found.

The low serial correlation and the apparent invariability of the $P(X_{n+1} > X_n)$ makes the use of sample paths and scatter plots of little value. Samples are provided in Figures III.E.4.1 through III.E.4.12.

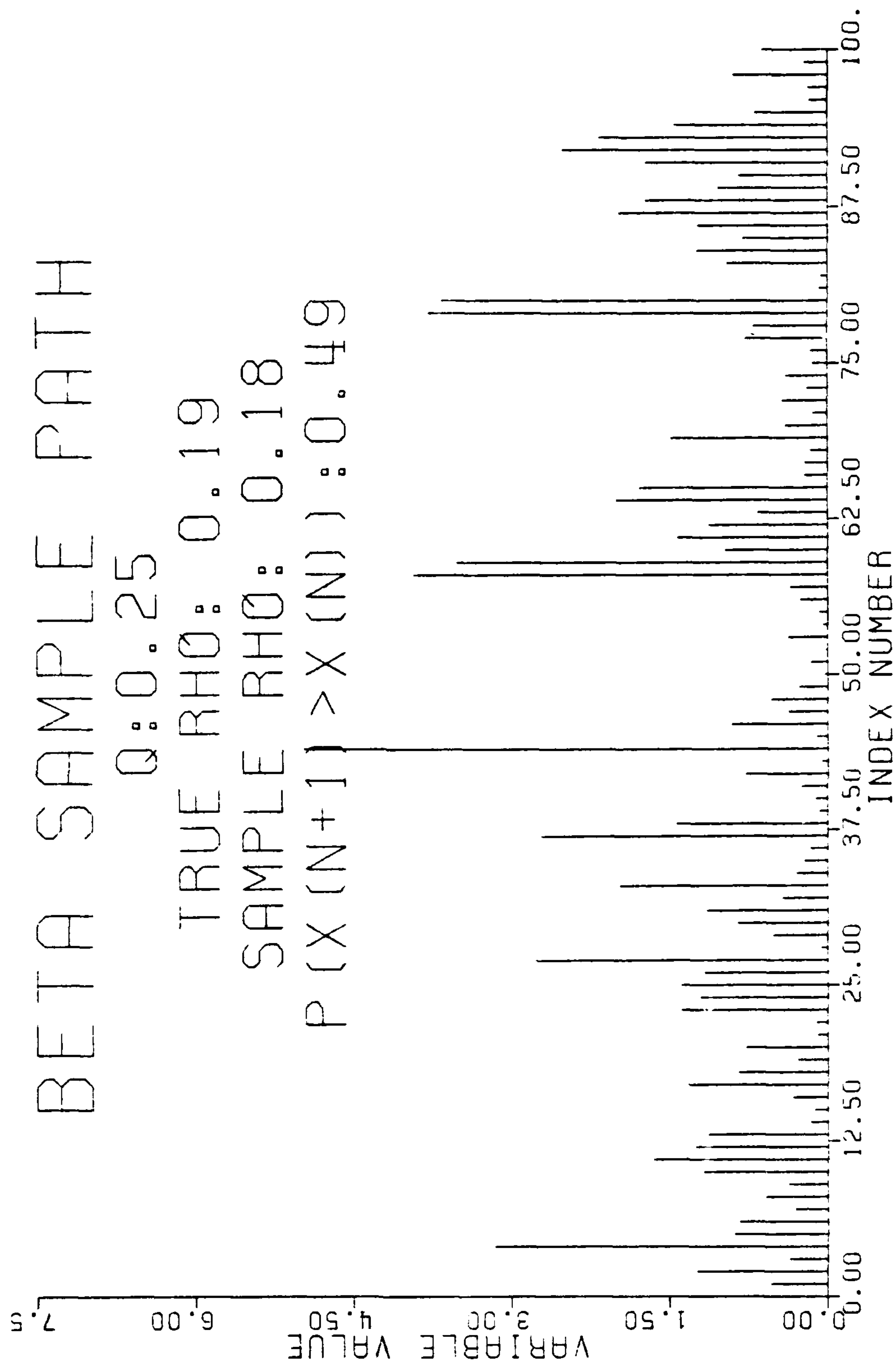


FIGURE III.E.4.1

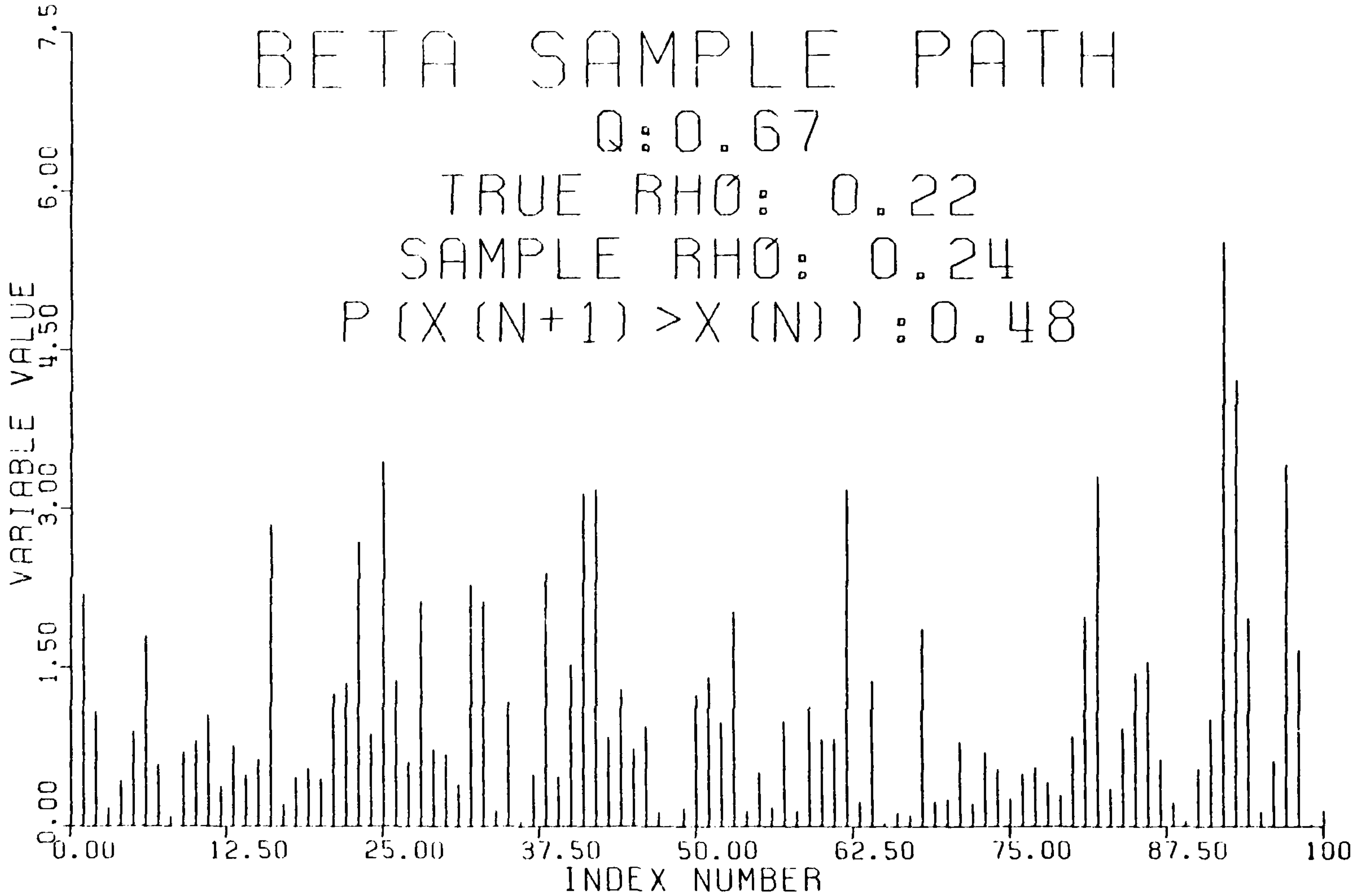


FIGURE III.E.4.2

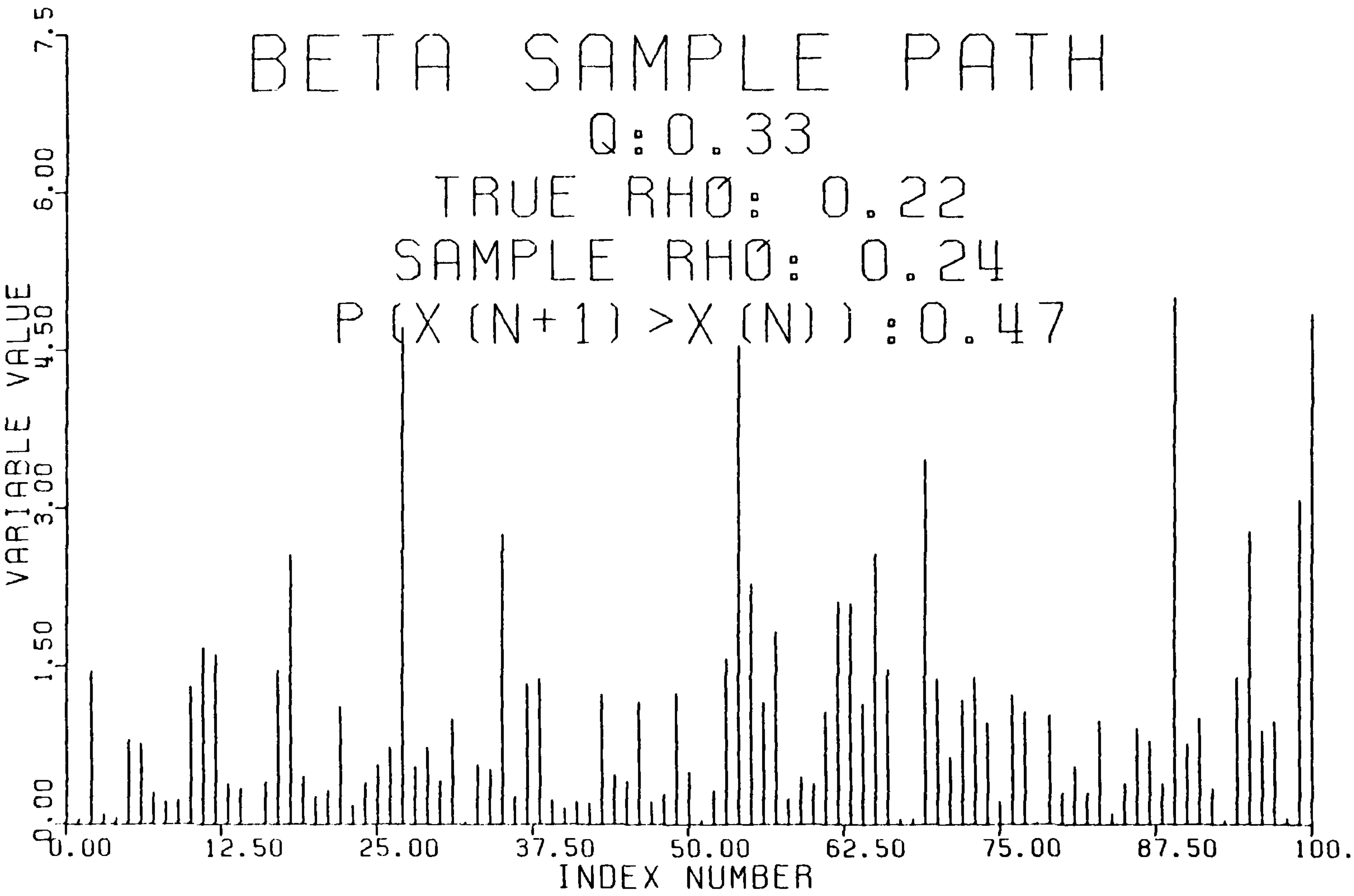


FIGURE III.E.4.3

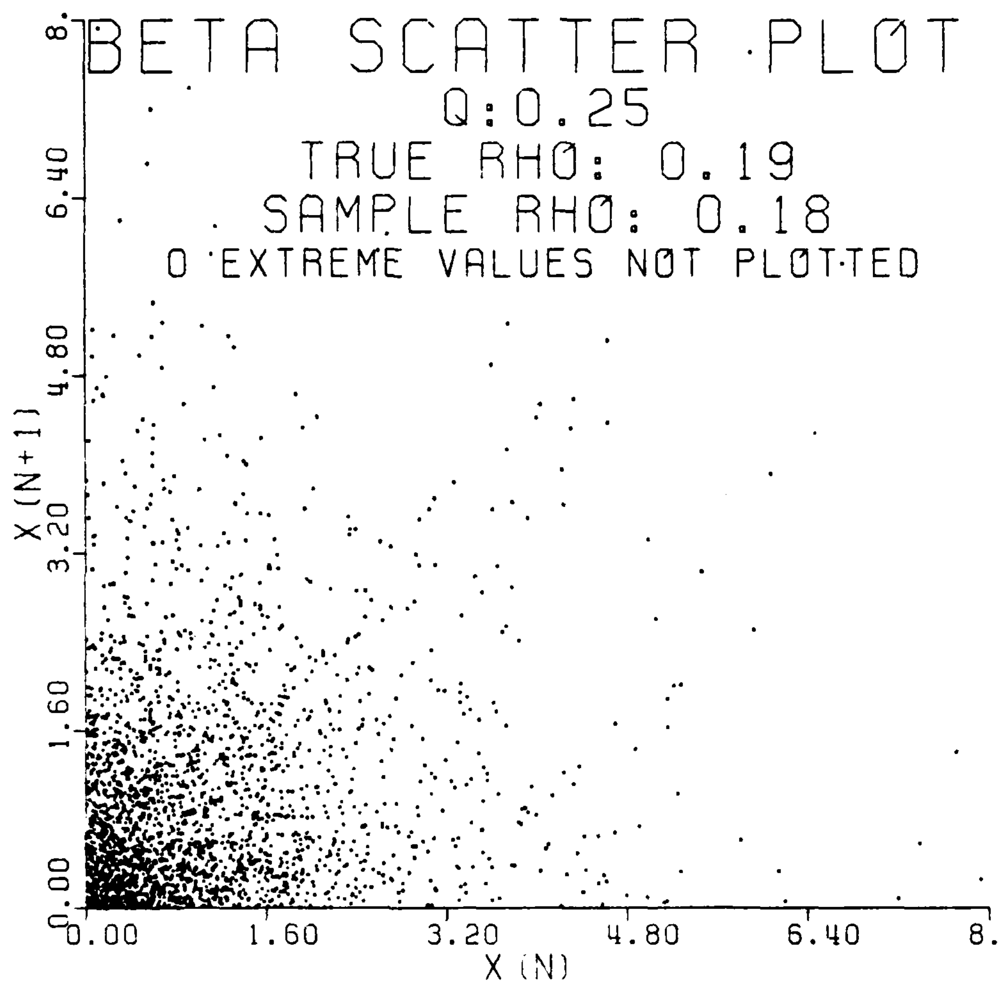


FIGURE III.E.4.4

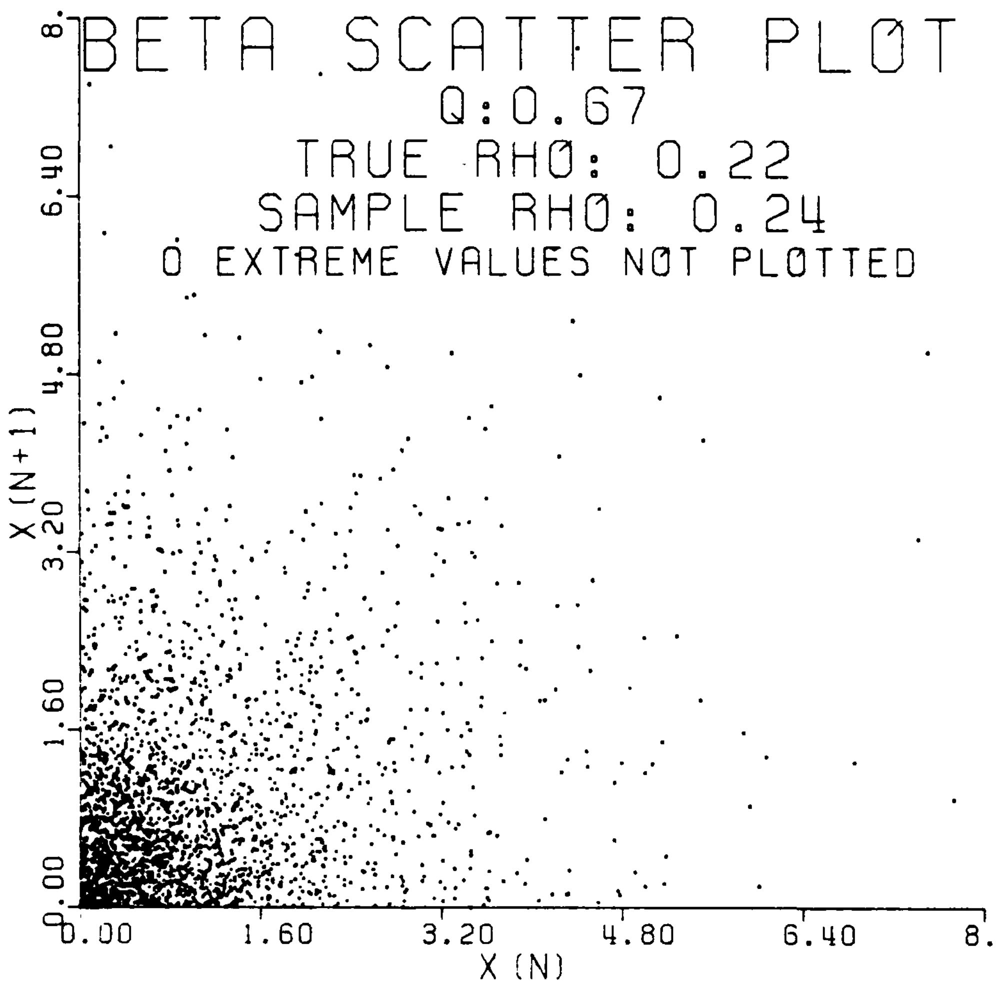


FIGURE III.E.4.5

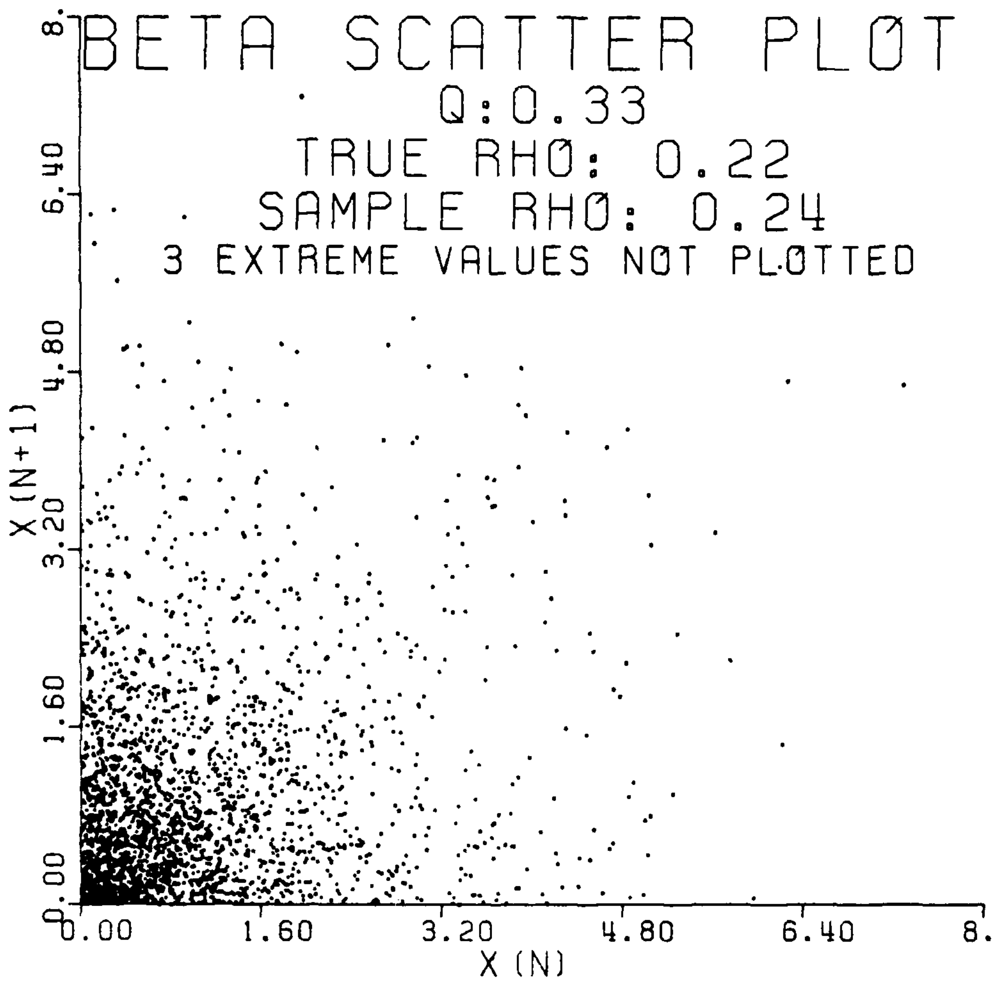


FIGURE III.E.4.6

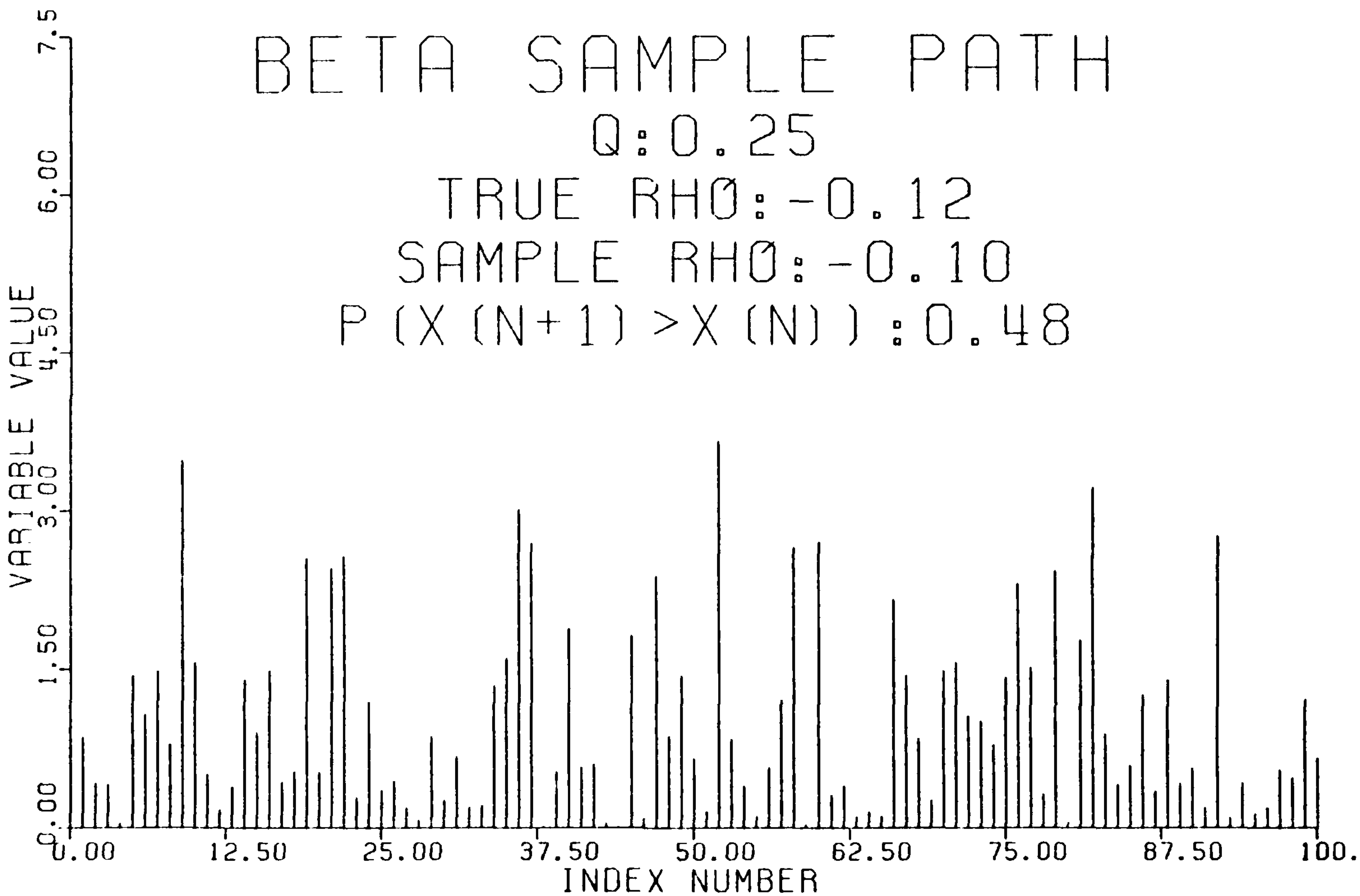


FIGURE III.E.4.7

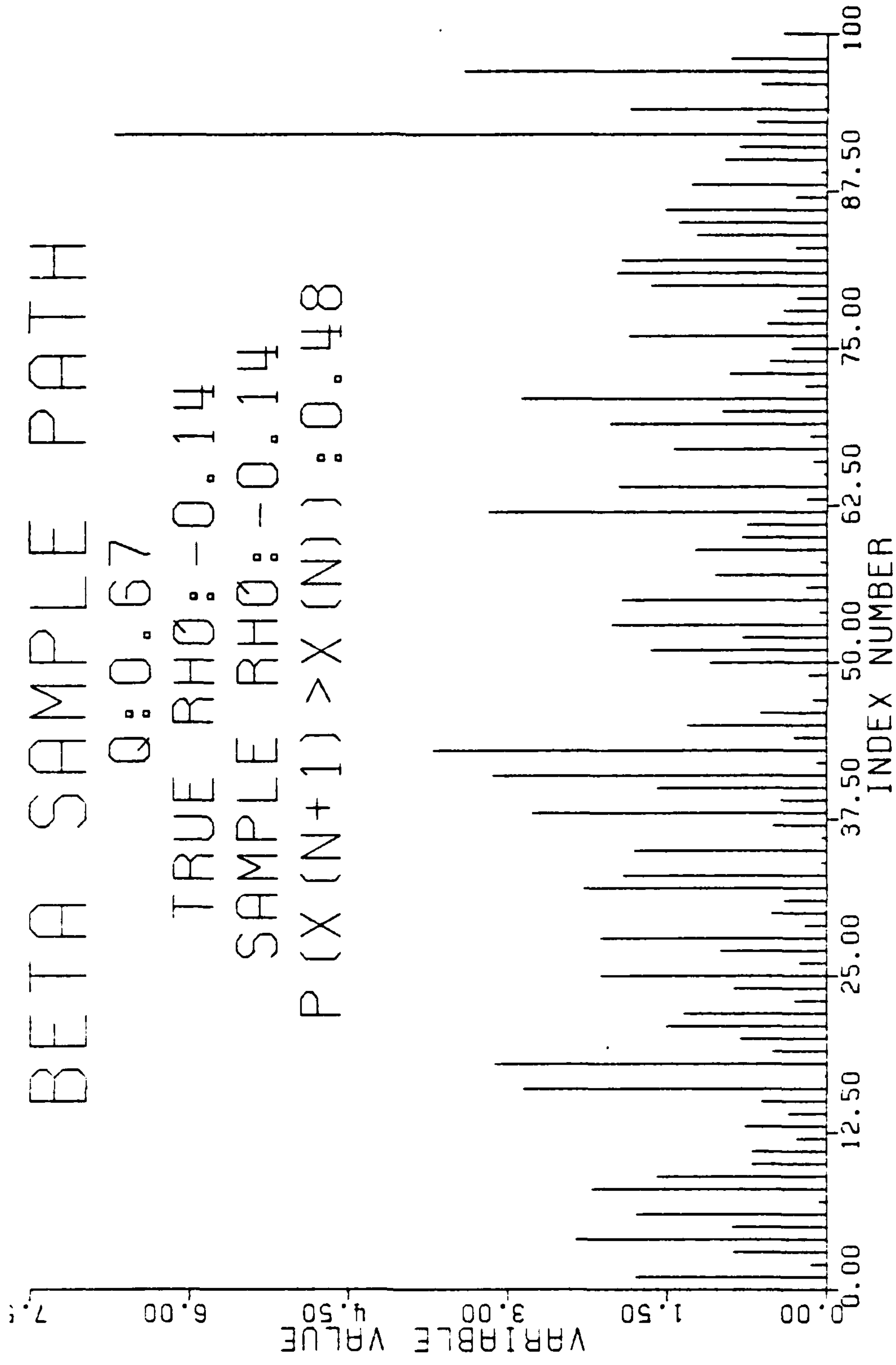


FIGURE III.E.4.8

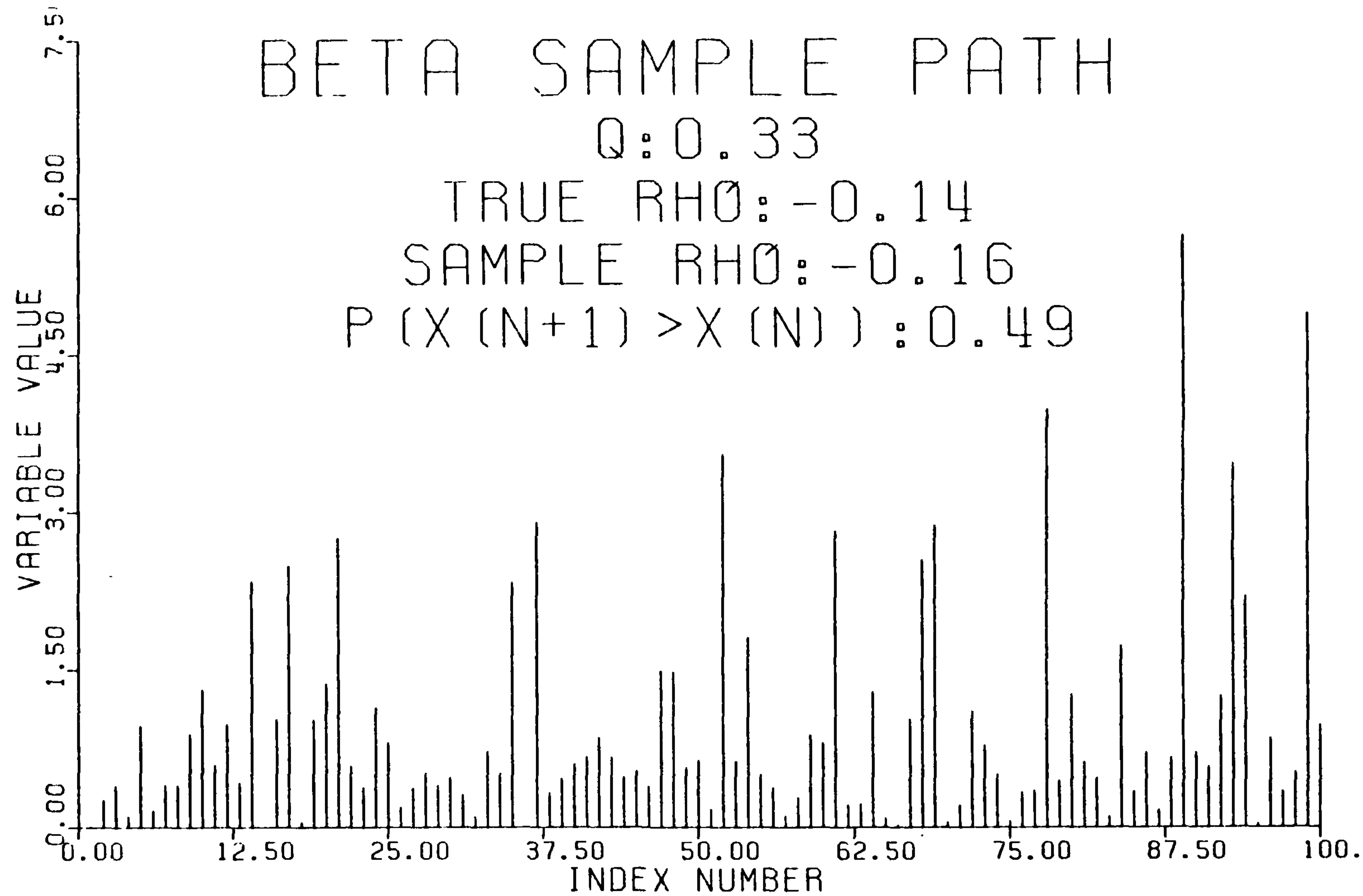


FIGURE III.E.4.9

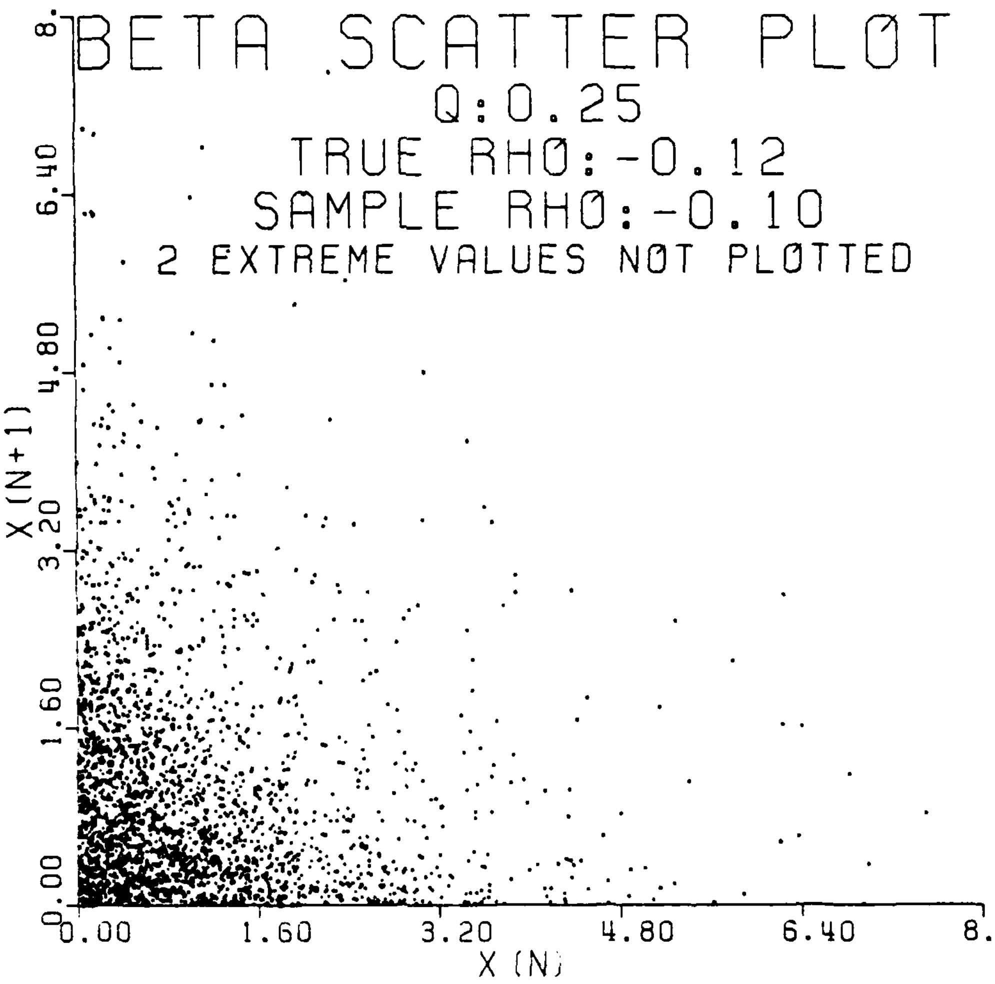


FIGURE III.E.4.10

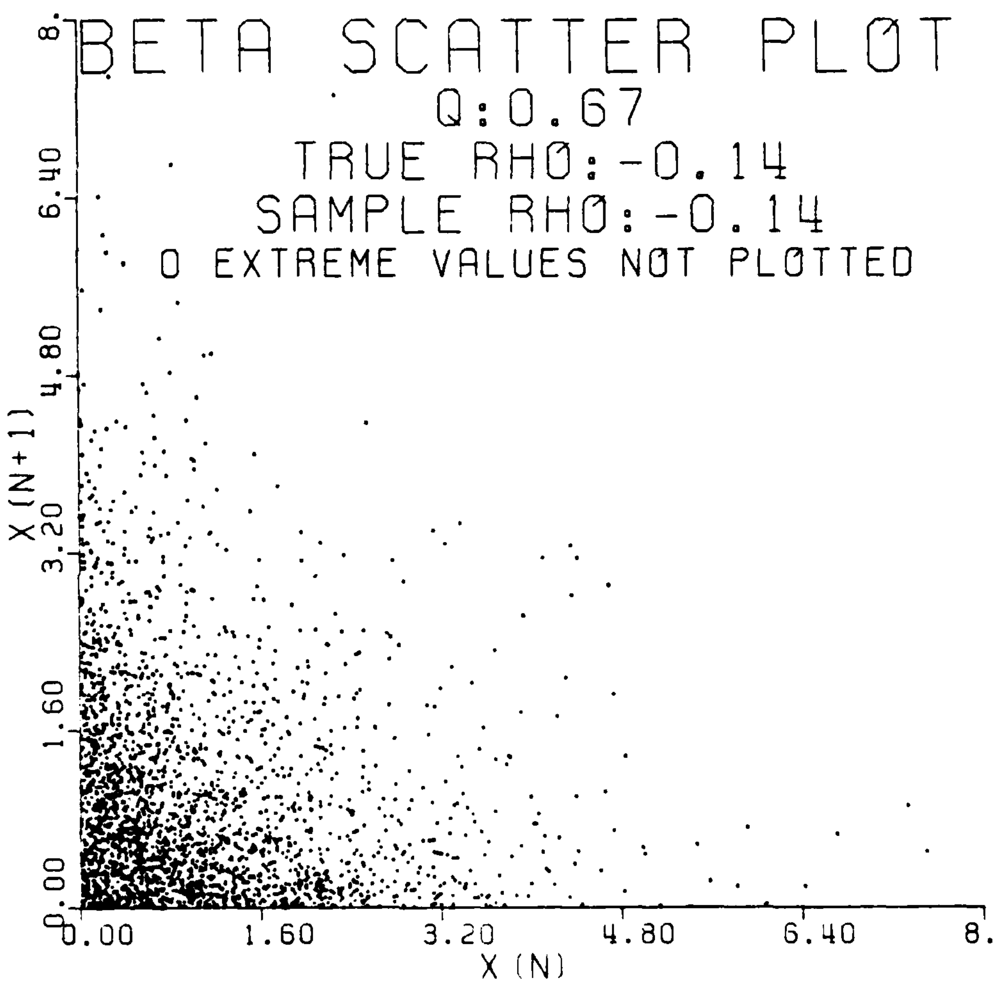


FIGURE III.E.4.11

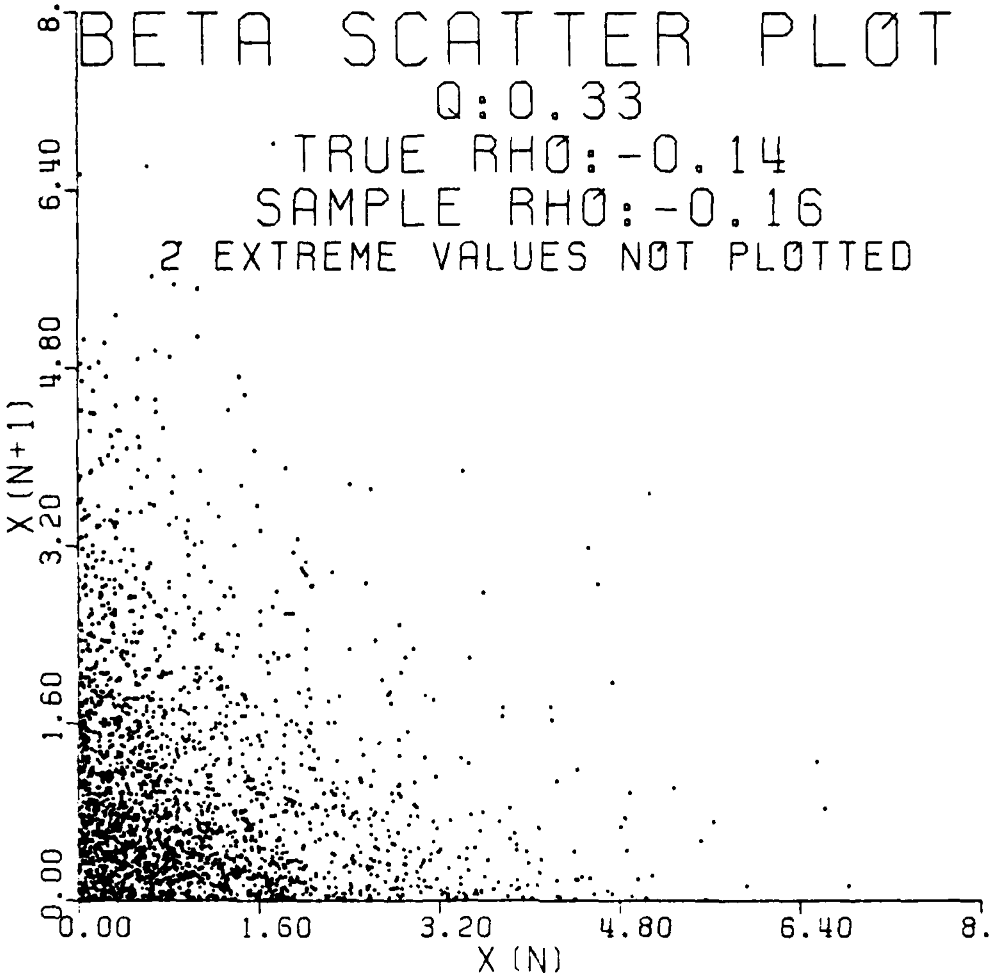


FIGURE III.E.4.12

IV. PRELIMINARY DATA ANALYSIS

A. INTRODUCTION

During the period 1955 through 1969 a weather ship stationed in the Gulf of Alaska (50°N,145°W) collected, among other data, wind speed data at three hour intervals [Ref. 33]. The existence of the wind speed data was brought to Professor Lewis' attention when a student in Oceanography asked him to provide a model suitable for simulating wind velocity data. The simulated data was required as input to models of ocean temperature mixing. A copy of fifteen years of wind speed data was obtained for this thesis. The intent was to do a preliminary data analysis and then determine whether any of the models discussed in this thesis could provide an adequate representation of this data and, hence, a method for generating wind velocity sample paths for oceanography simulation.

Initially, the models discussed here are strong a priori candidates for data of this nature. Intuitively, there is a strong feeling that an assumption of independence among the data is unwarranted. Hence, autoregressive and moving average models which are designed to reflect the behavior of correlated data should be considered likely candidates. The non-negative nature of the data mitigates against the use of classical time series anlaysis which is based on the assumption of a normal distribution, and hence negative values, for the series. The existence of zeros in the data tends to make

the use of transformations, like taking the log, somewhat less appealing than otherwise. These considerations indicate that the models discussed in this thesis should be considered as likely candidates for modeling the wind speed data.

B. ANALYSIS OF THE RAW DATA

There were 43,800 data points available for analysis, 2920 for each of the fifteen years between 1955 and 1969 inclusive (the extra data for leap years was discarded). Since this size data base made it inconvenient, if not impossible, to manipulate by hand, each year's data was plotted as a means to promote familiarity with the data. The plot of each year's data and the plot of the data averaged over all fifteen years (e.g., all data taken at 0300 on 1 January of each year were averaged) are presented in Figures IV.B.1a through IV.B.1p. Several characteristics can be observed from these figures. First, and perhaps most obvious, is the expected yearly cycle of the data. Values at the beginning and end of the year tend to be higher than those in the middle. Second, the data is discretized to a large extent. There exist obvious horizontal lines of equal valued data. A sort and plot of the entire data set reveals that the data consists of values that are integral multiples of 1.03 with a few values between these multiples. Next, on some occasions a series of high values will all be equal,

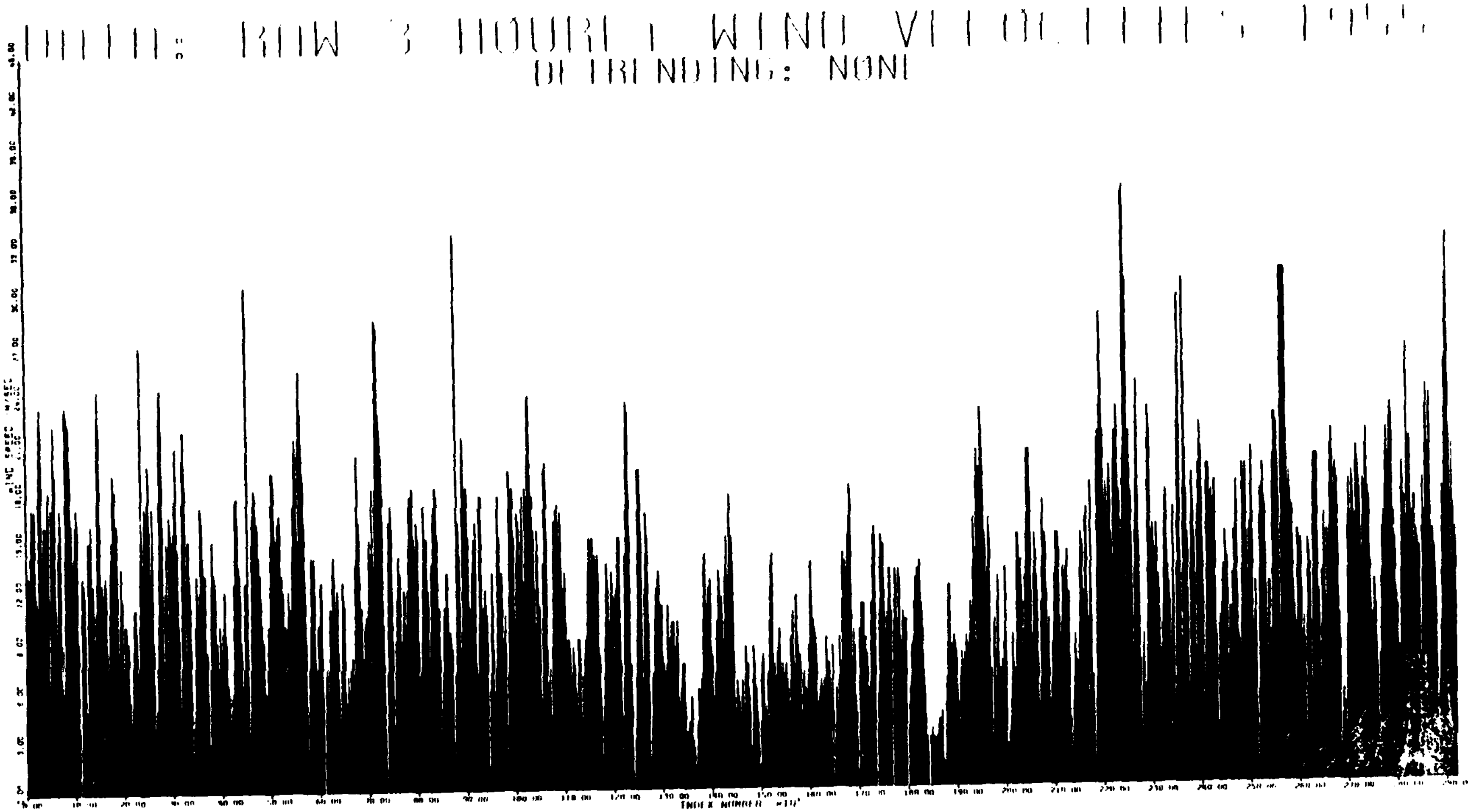


Figure IV.B.1a. 1955 Raw data.

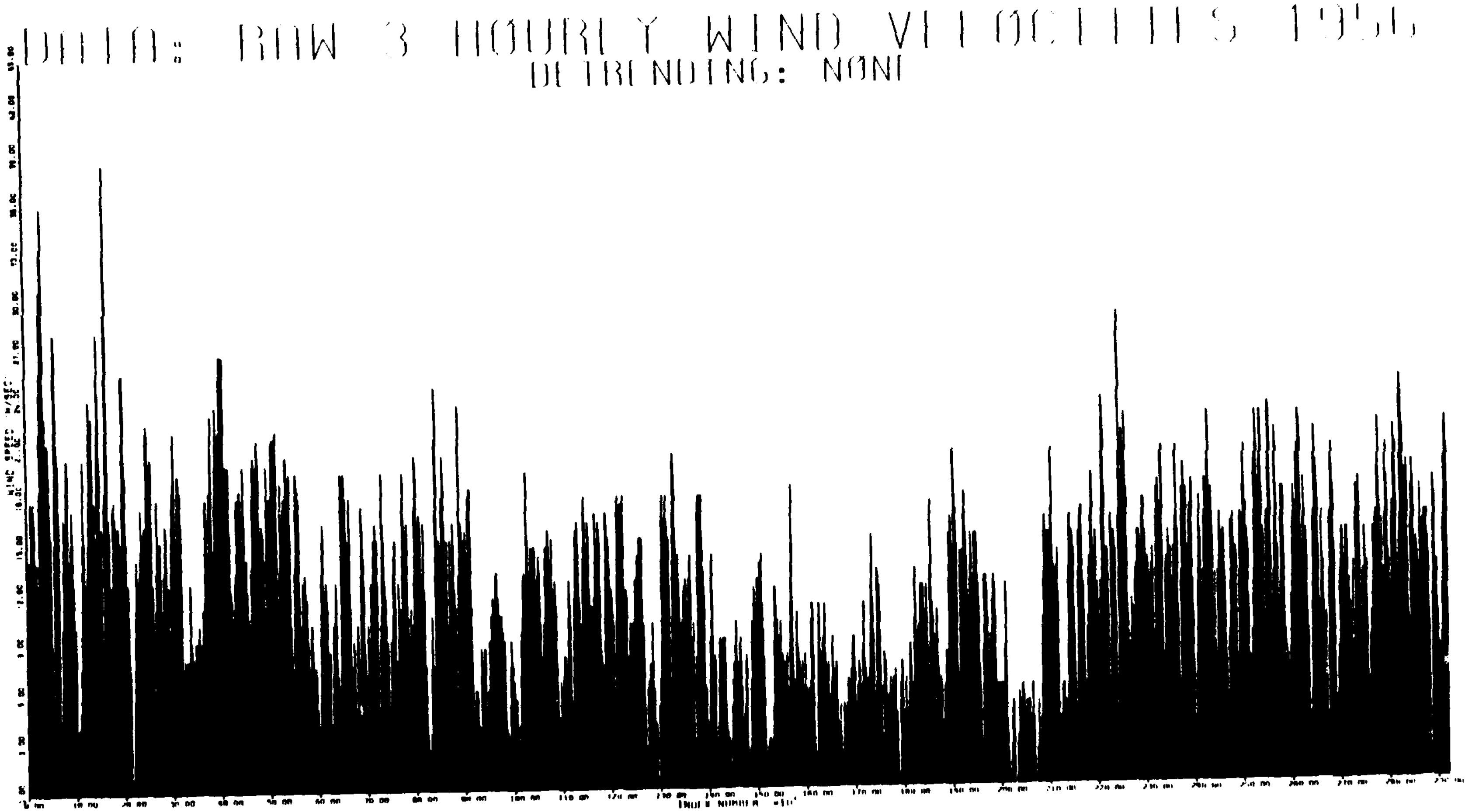


Figure IV.B.1b. 1956 Raw data.

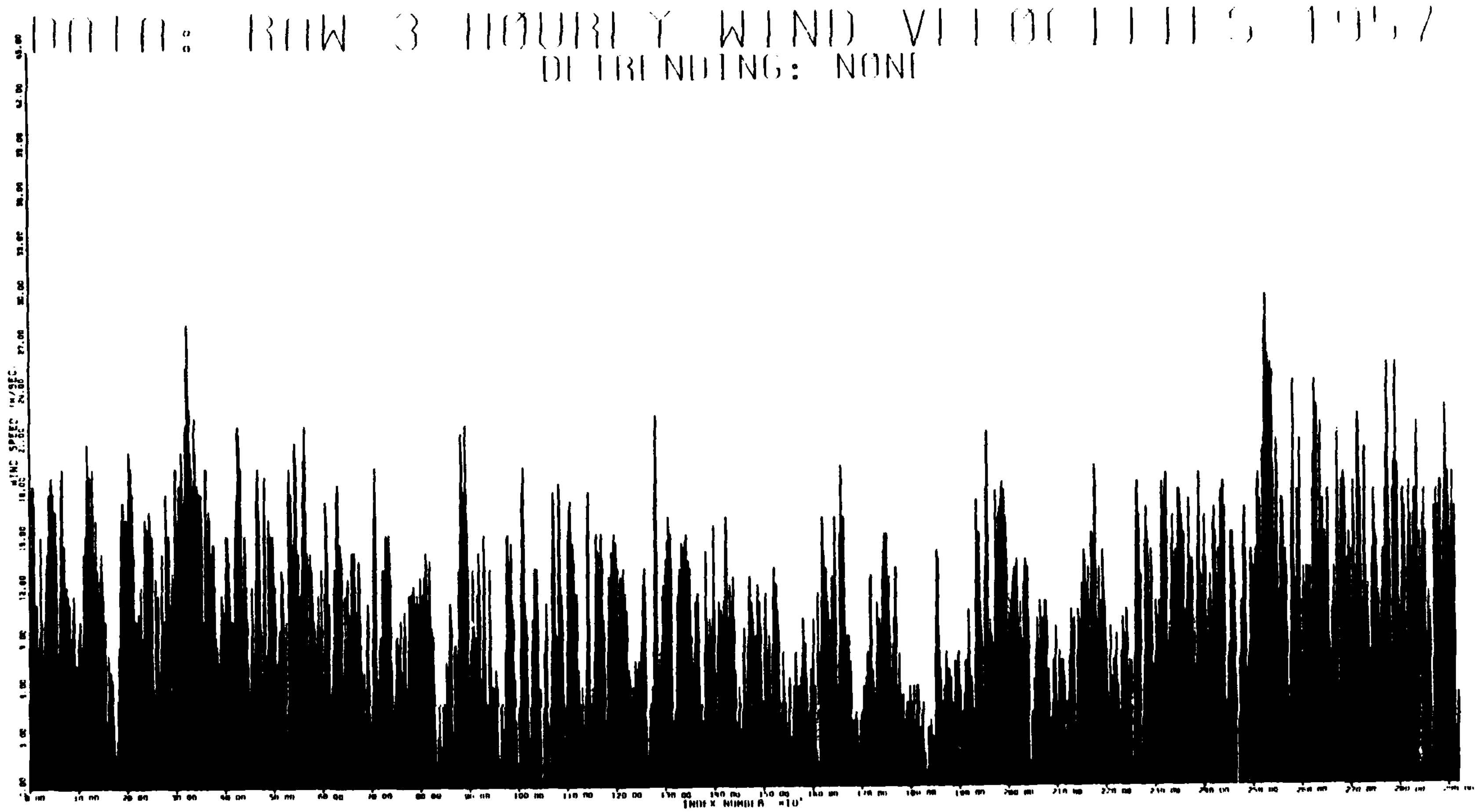


Figure IV.B.1c.   1957 Raw data.

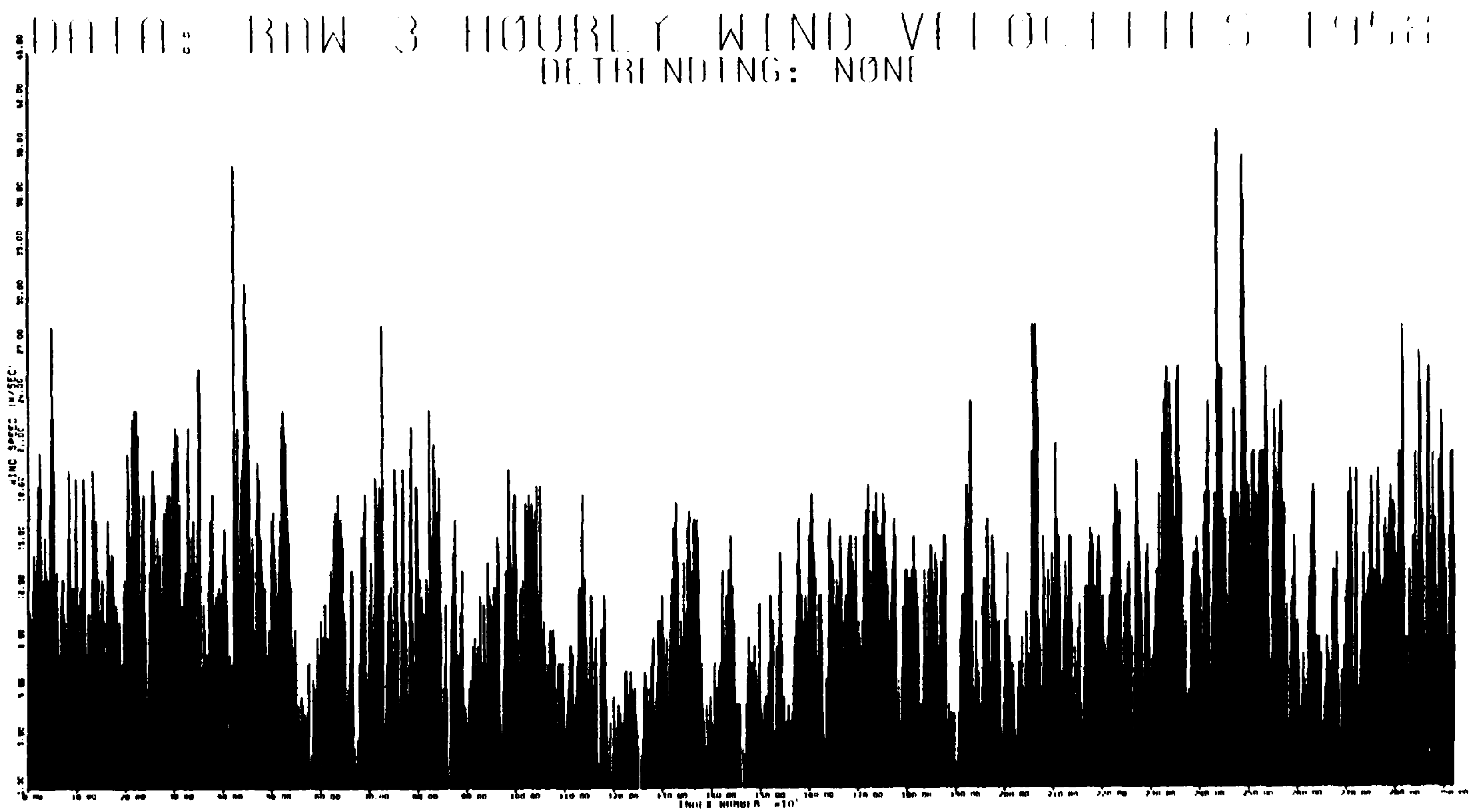


Figure IV.B.1d.   1958 Raw data.

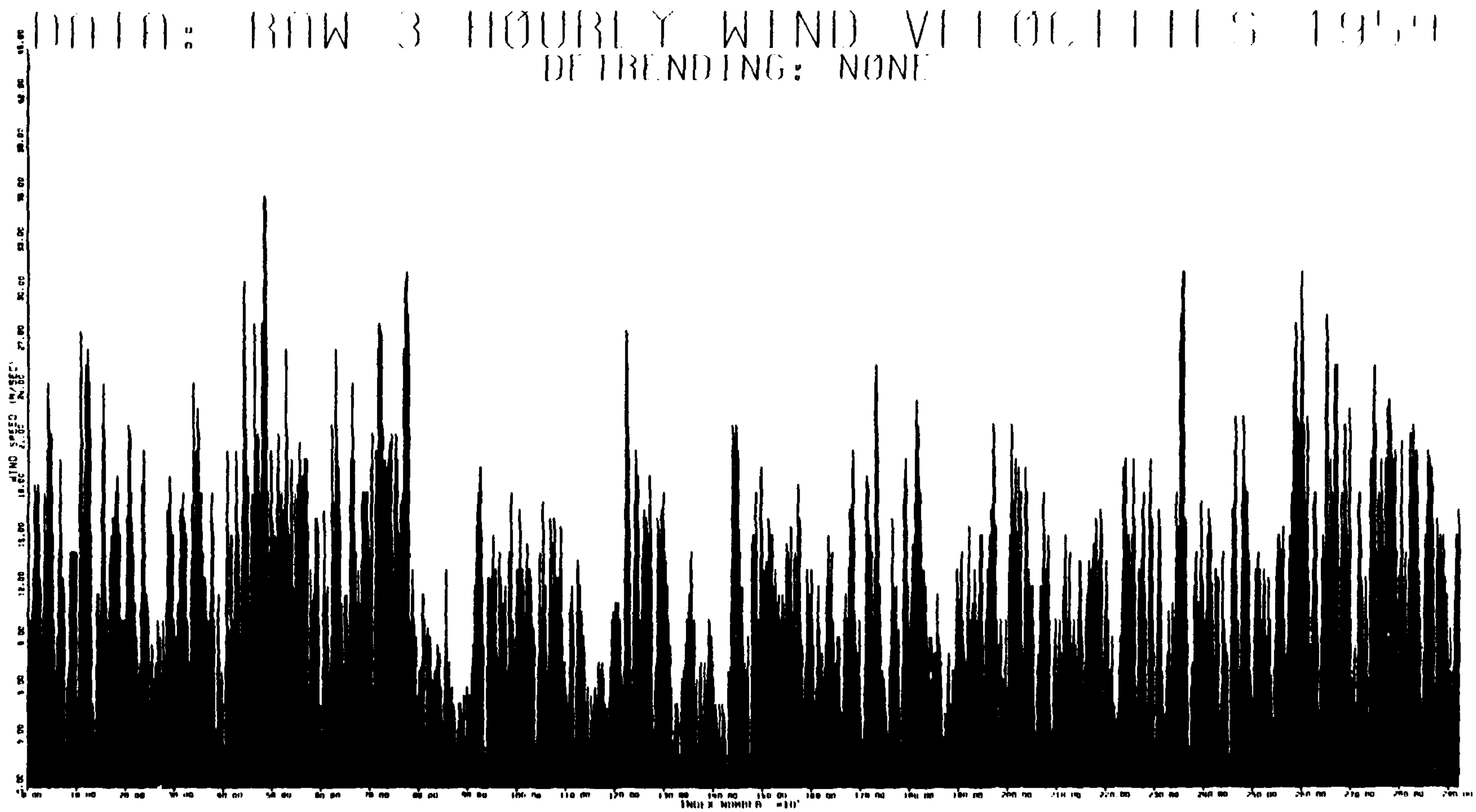


Figure IV.B.1e. 1959 Raw data.

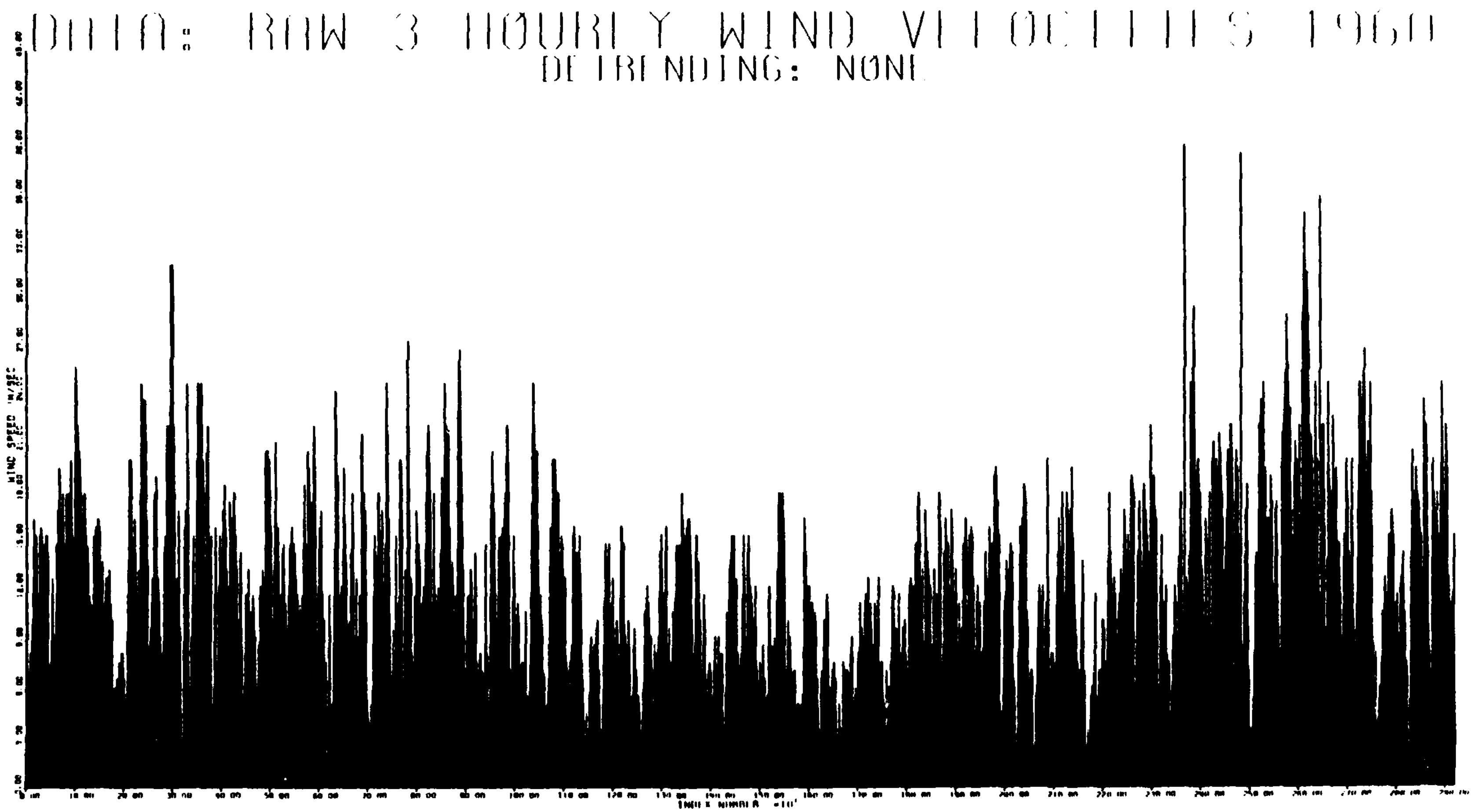


Figure IV.B.1f. 1960 Raw data.

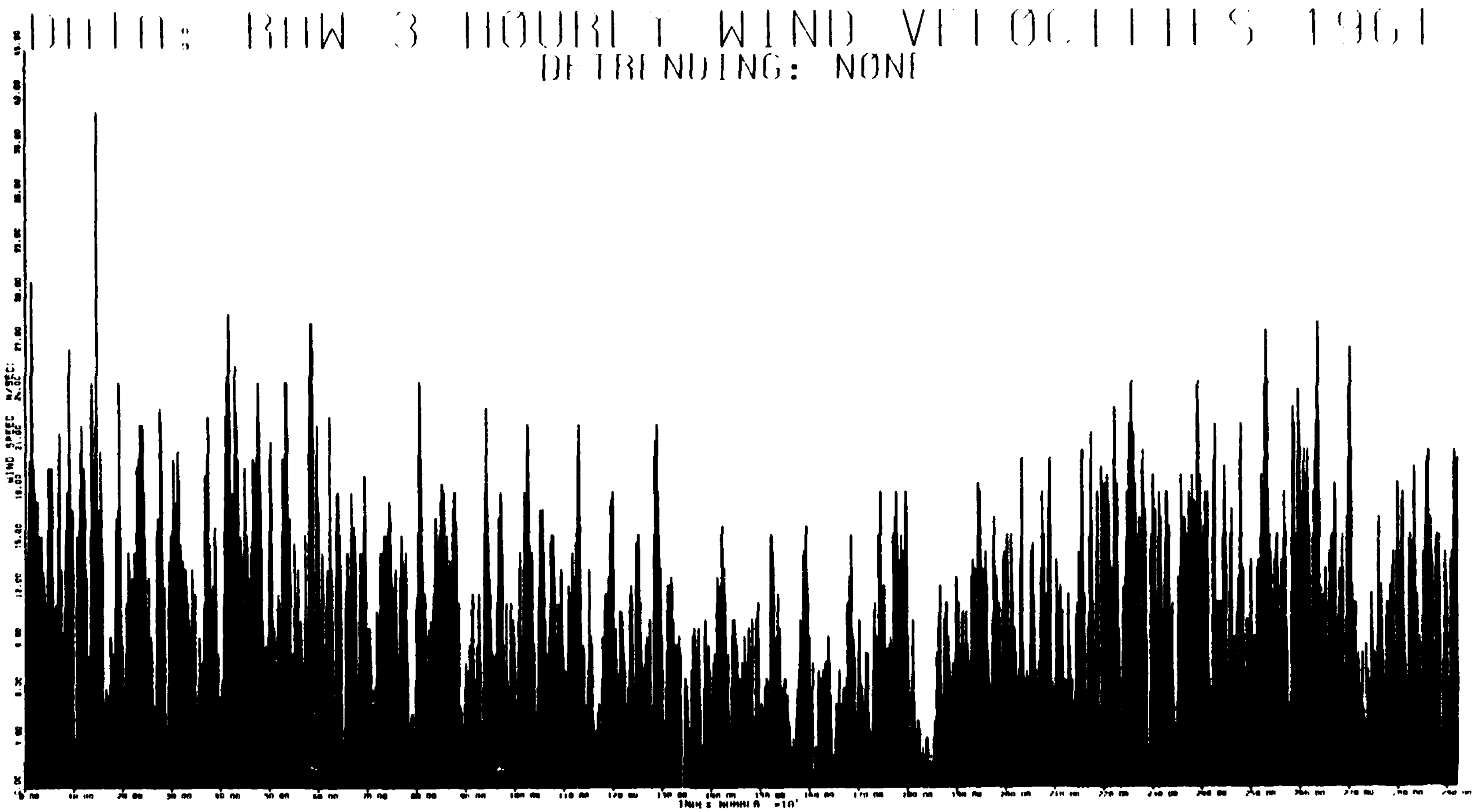


Figure IV.B.1g. 1961 Raw data.

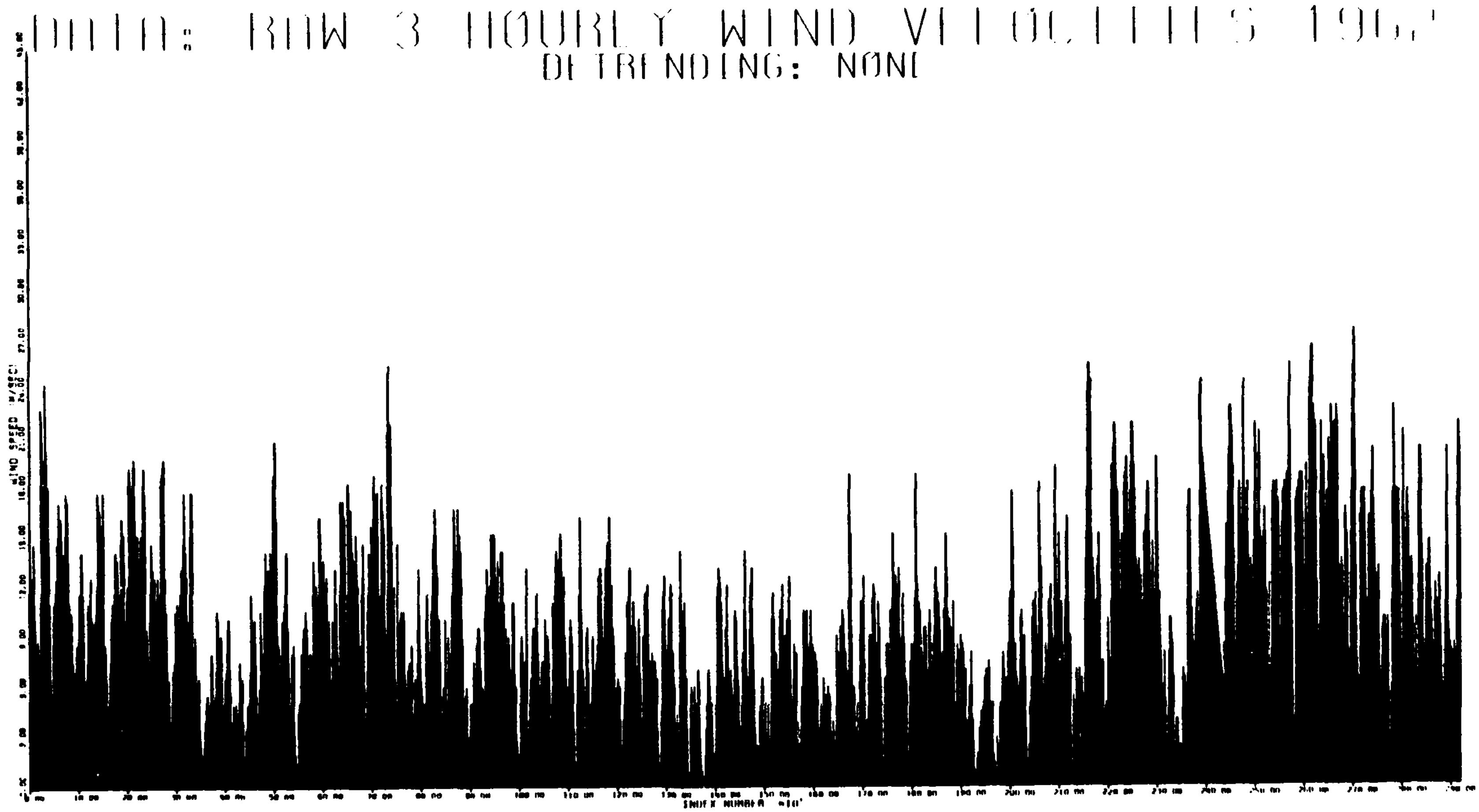


Figure IV.B.1b. 1962 Raw data.

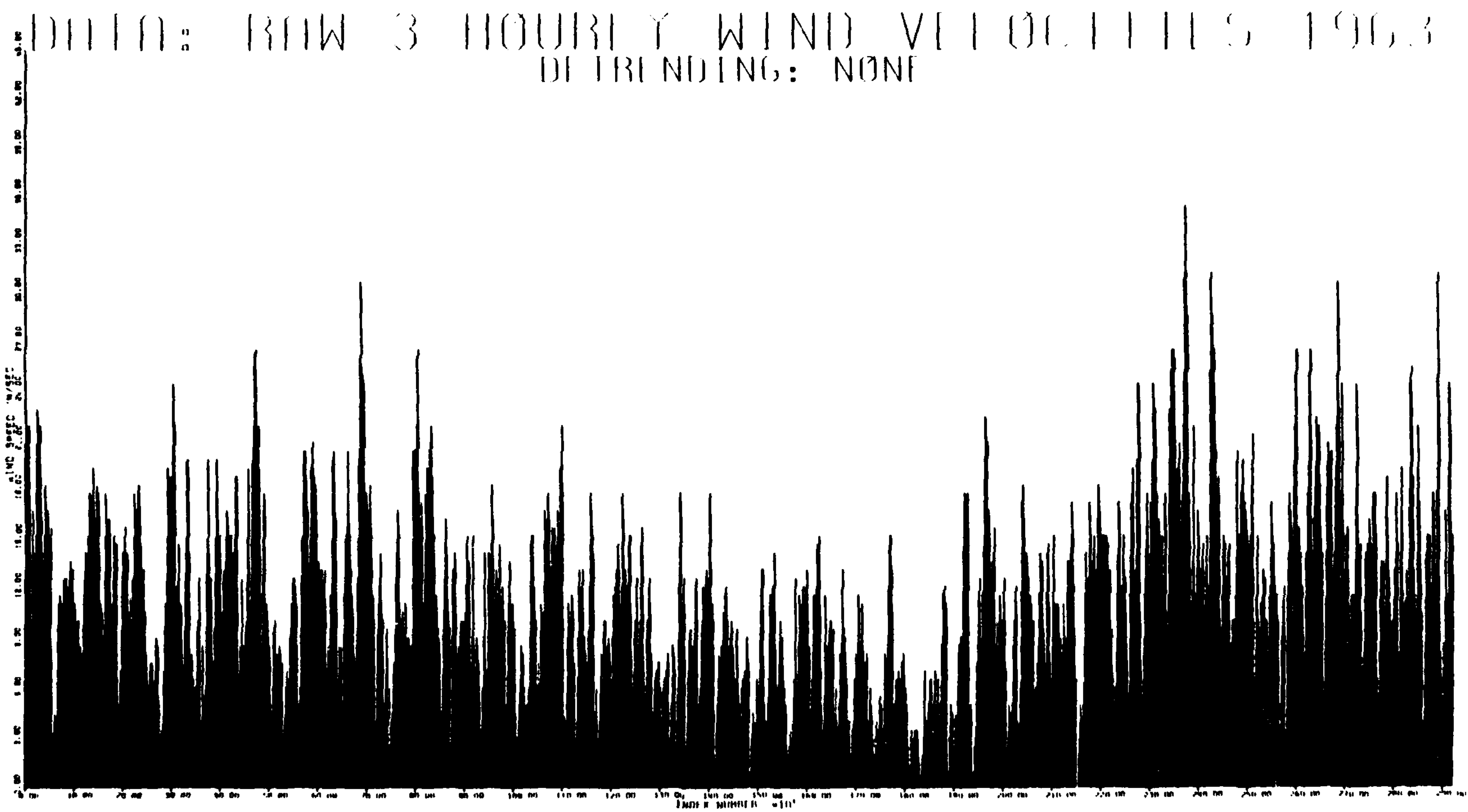


Figure IV.B.1i. 1963 Raw data.

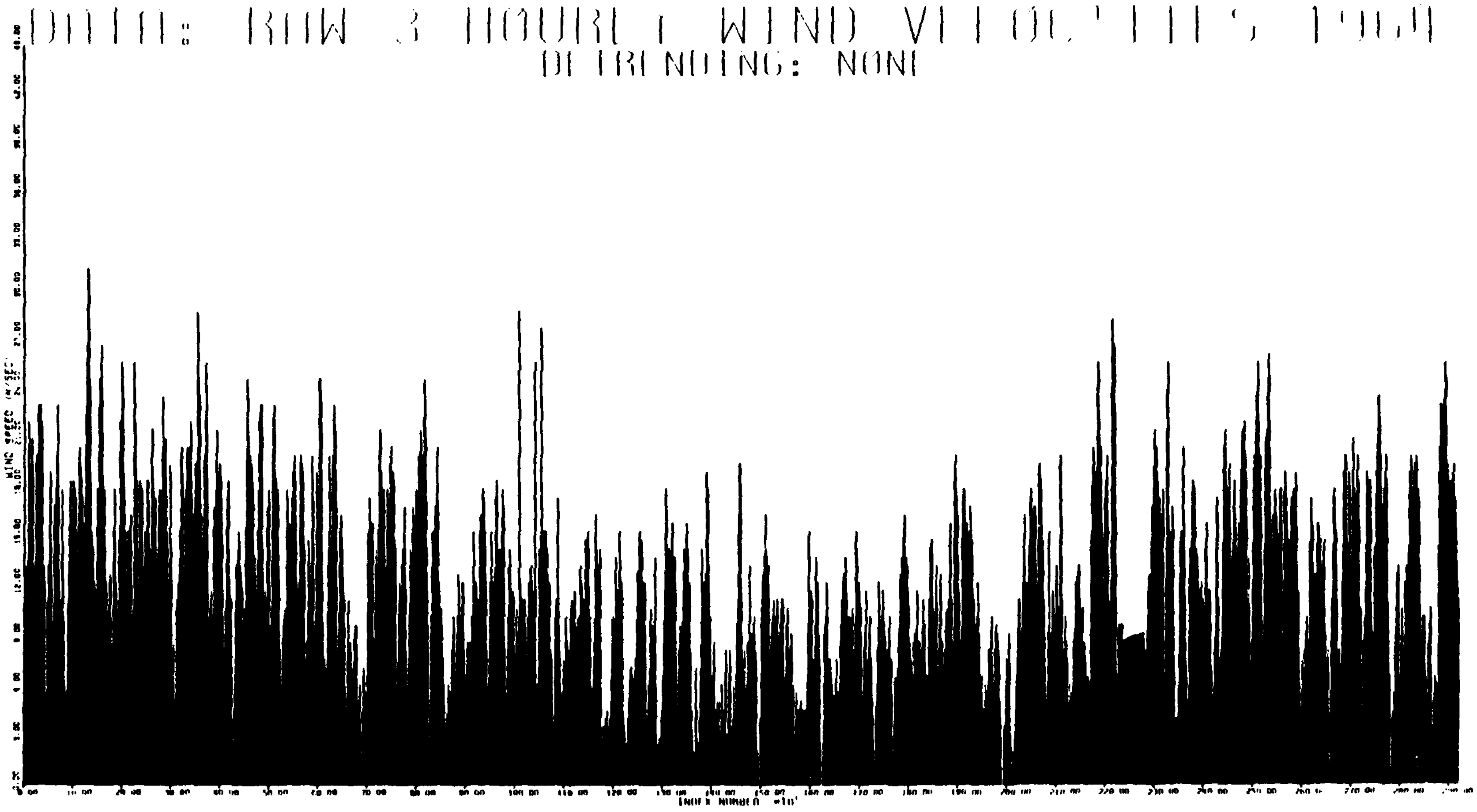


Figure IV.B.1j). 1964 Raw data.

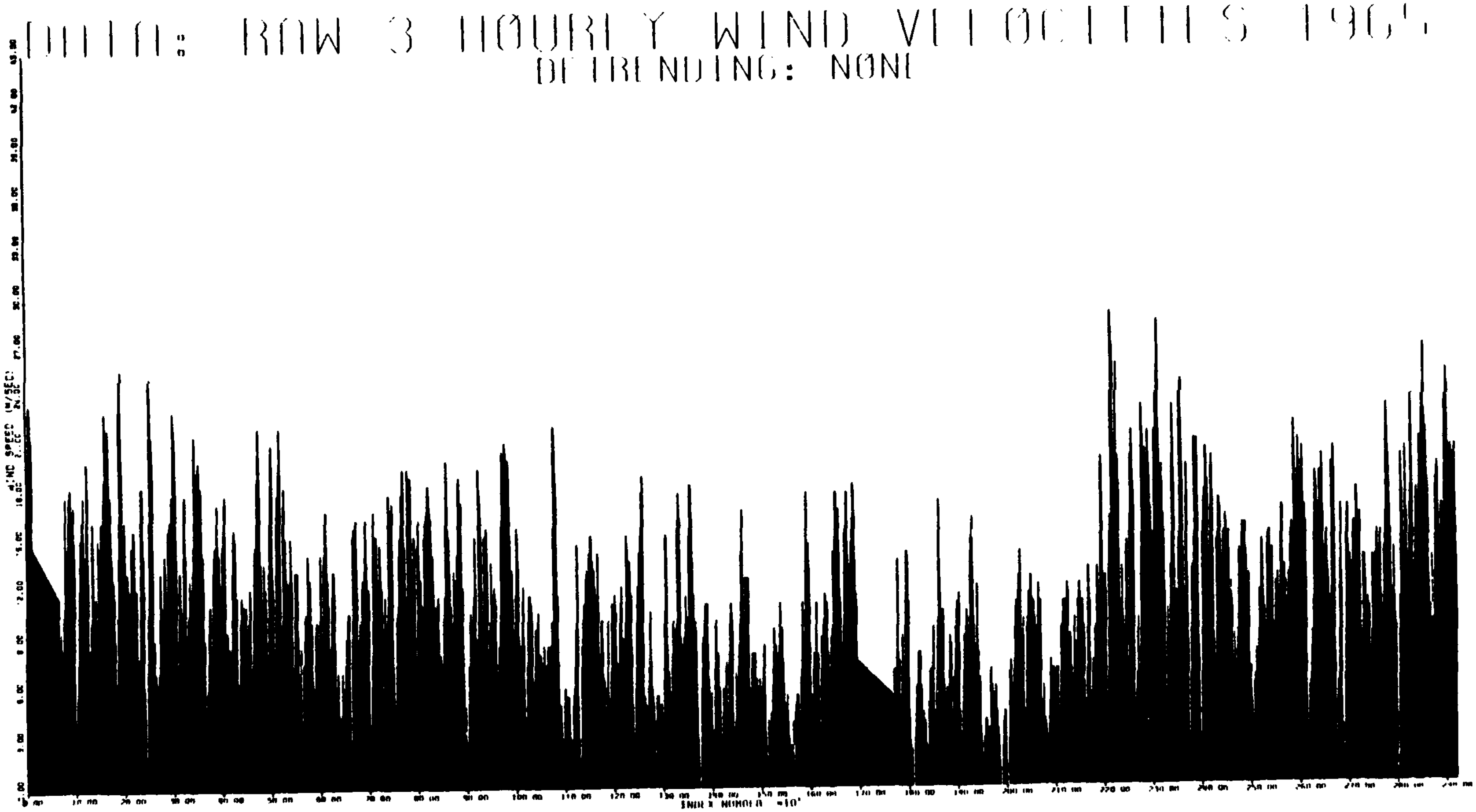


Figure IV.B.1k. 1965 Raw data.

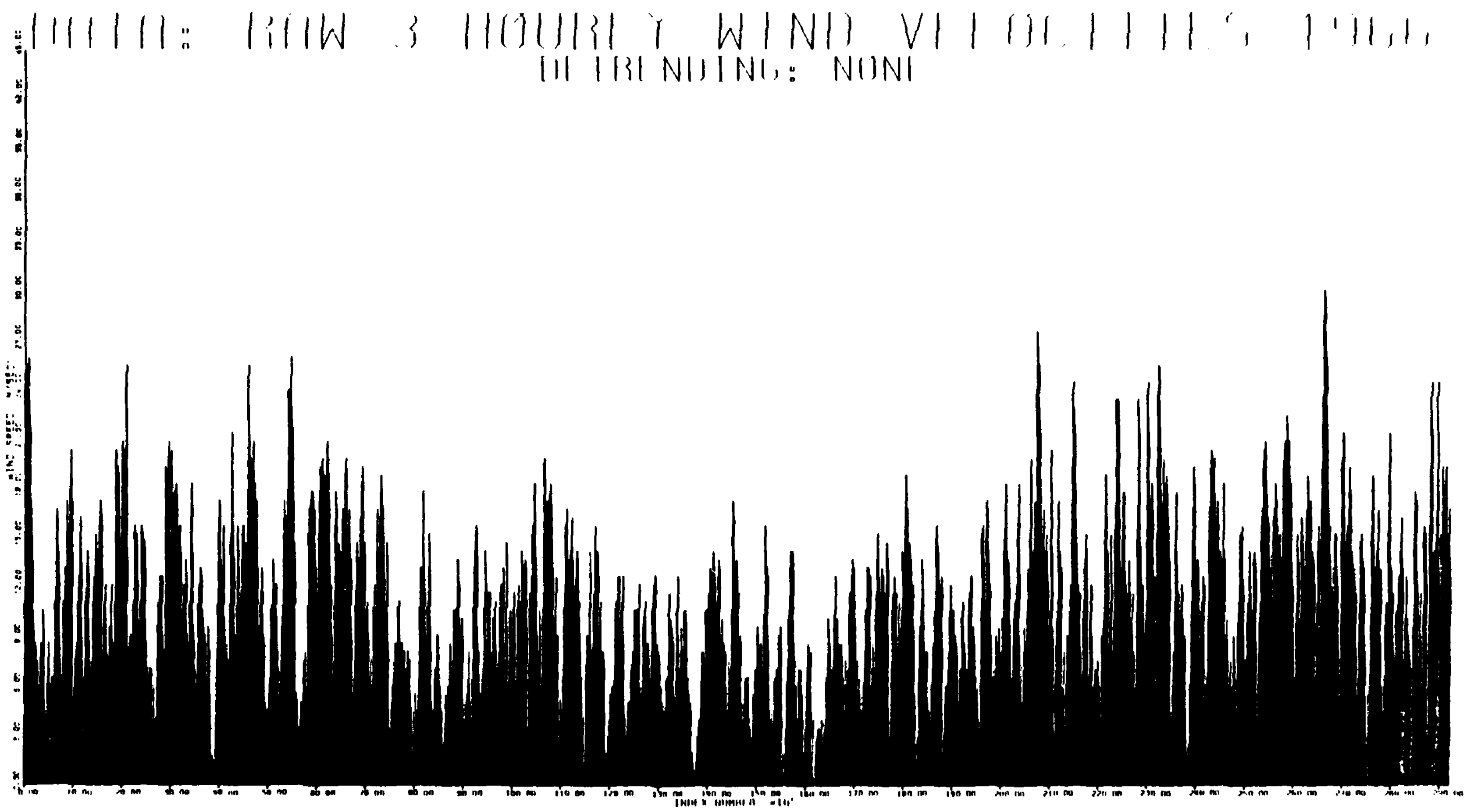


Figure IV.B.11. 1966 Raw data.

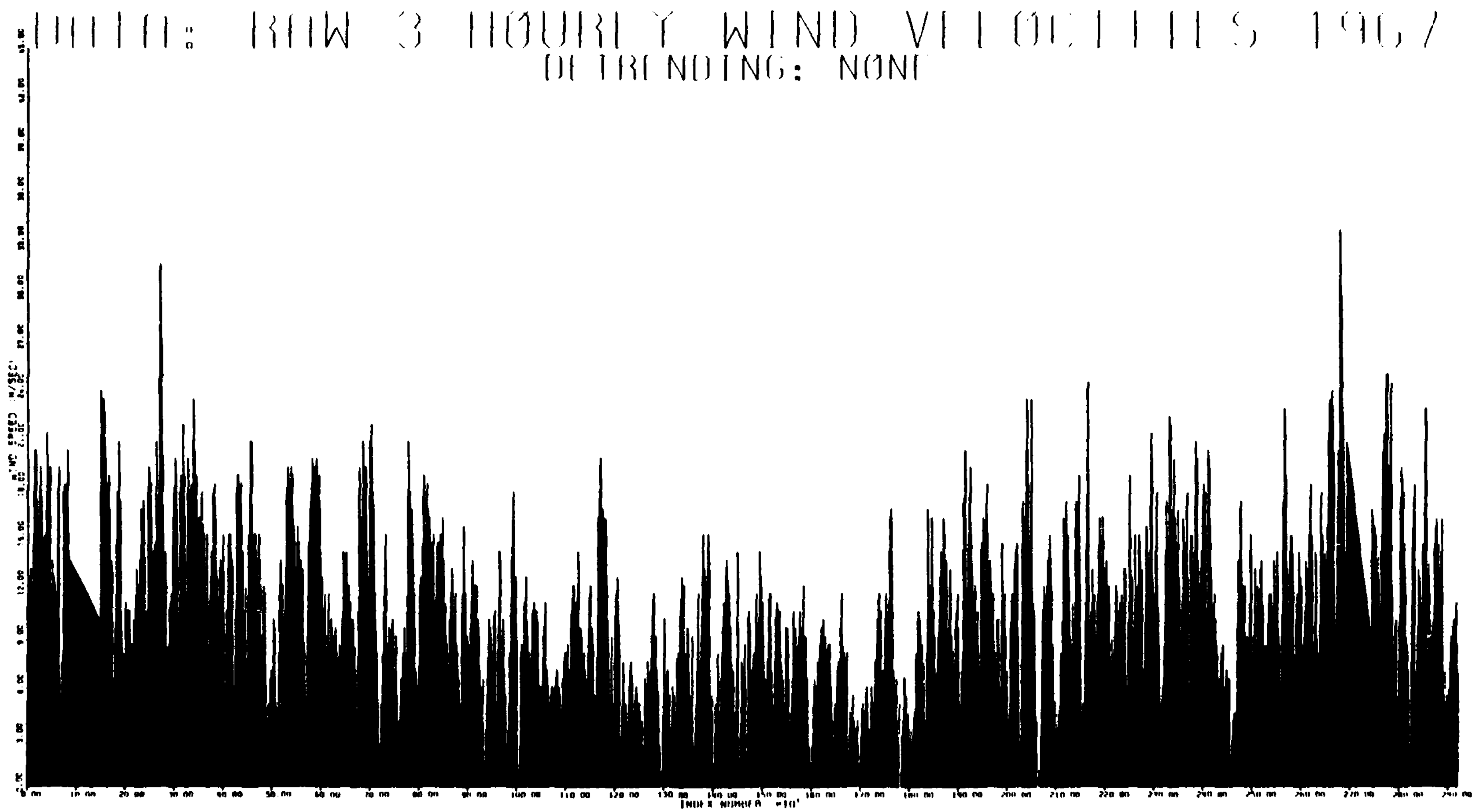


Figure IV.B.1m. 1967 Raw data.

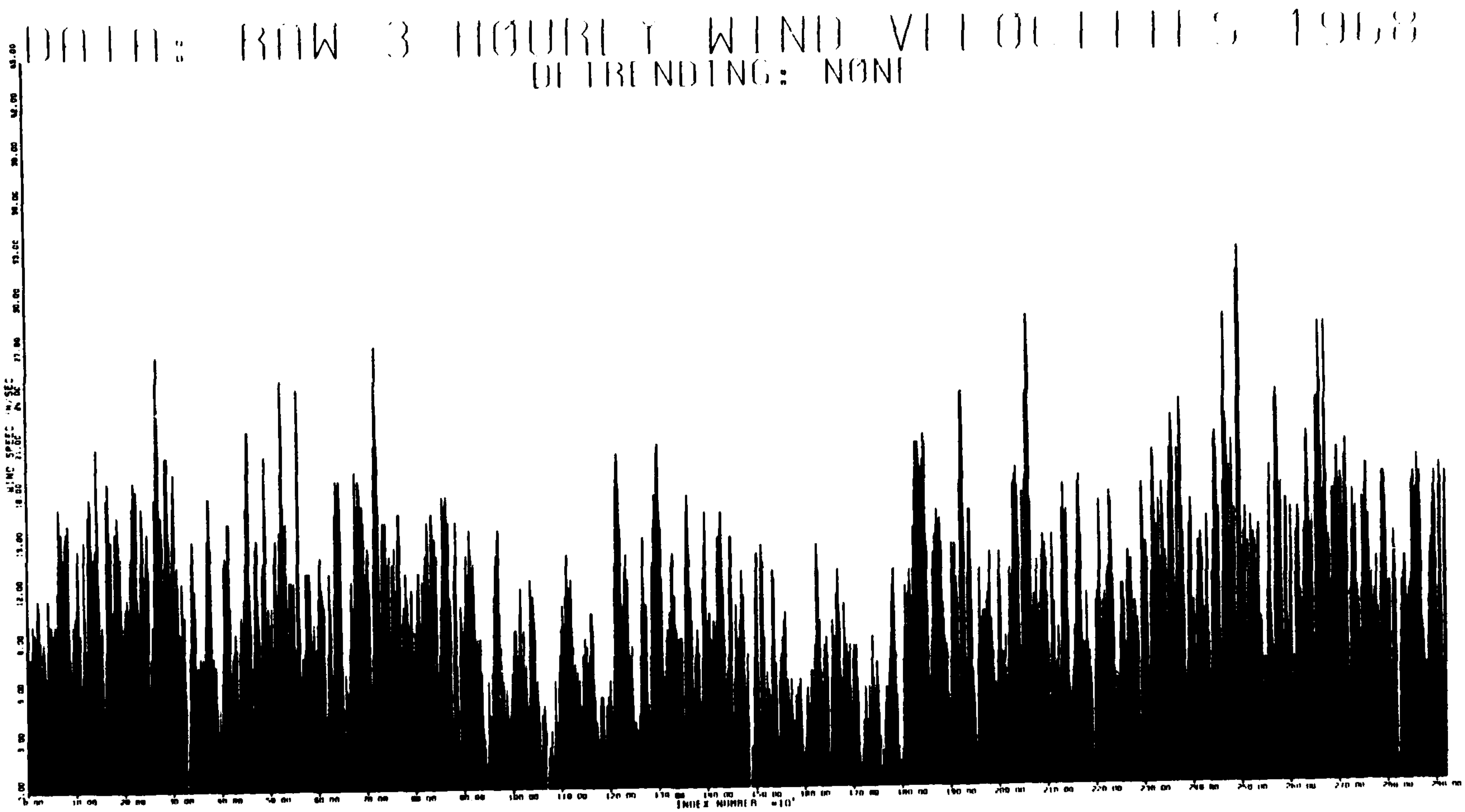


Figure IV.B.1n.   1968 Raw data.

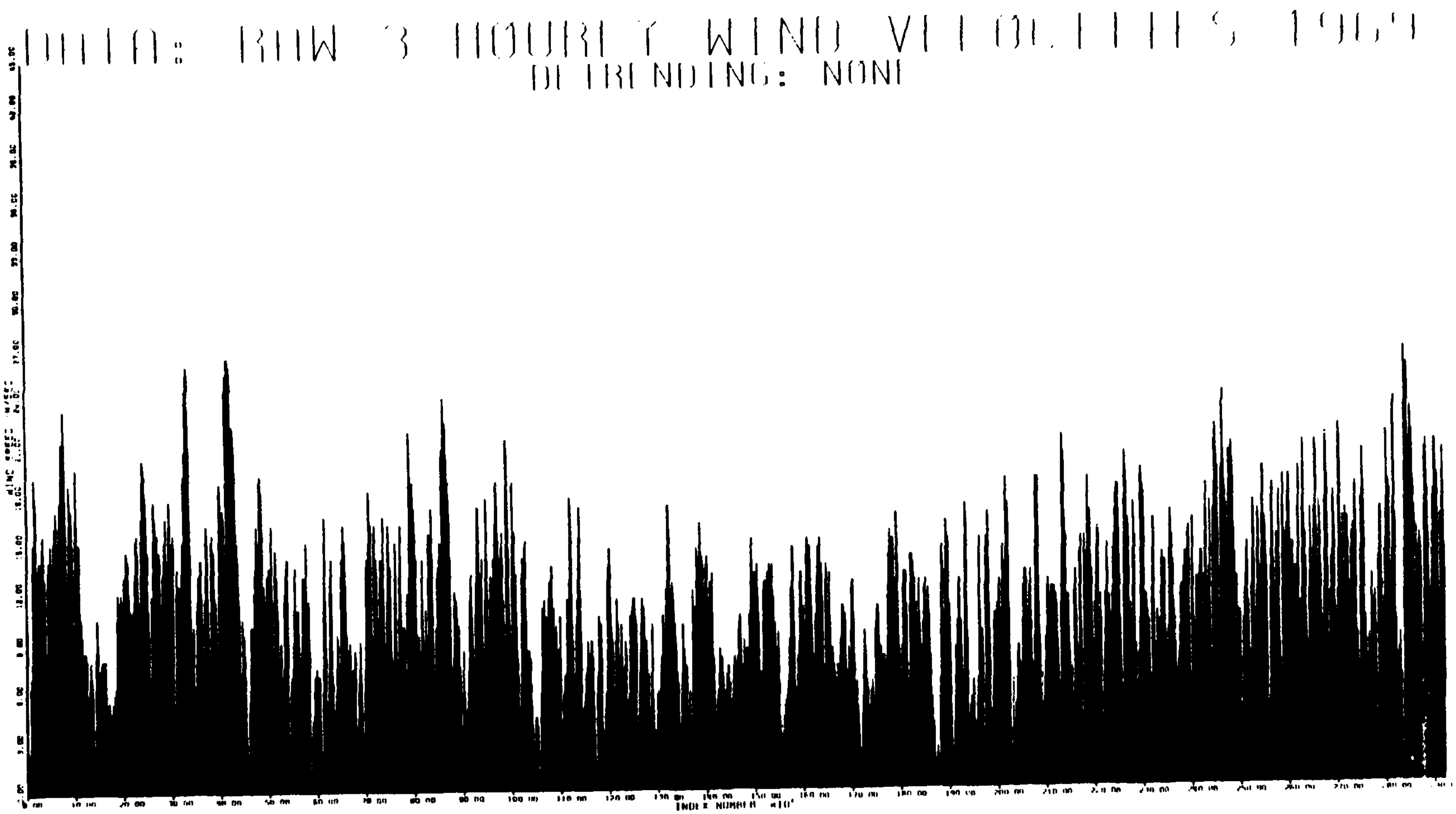


Figure IV.B.10. 1969 Raw data.

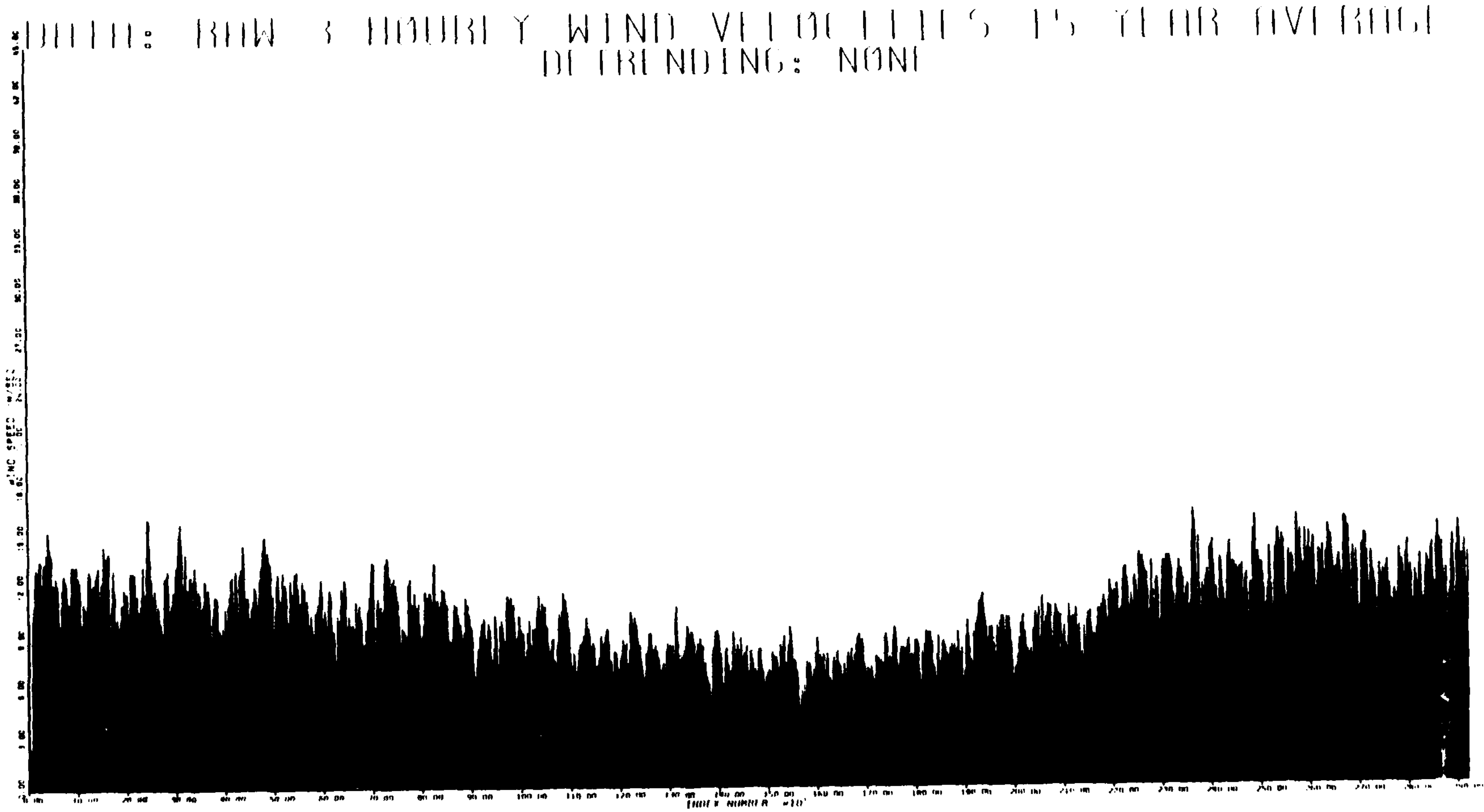


Figure IV.B.1p. 15 year average raw data. This shows the yearly cycle clearly.

indicating that some clipping may have occurred (see Figure IV.B.1a vicinity of 2575 and 2625, Figure IV.B.1b vicinity of 400 and 525, etc.). These last two characteristics indicate that statistical properties which are sensitive to the behavior of the "tail" of a distribution may be affected. The final observation about the data is that there are apparently intervals when the data was not actually collected. These instances appear as reasonably long strings of values which have a strong linear appearance (as though the values were produced by linearly interpolating between two boundary values). See Figures IV.B.1h (vicinity 2400), IV.B.1j (vicinity 2250), IV.B.1k (vicinity 50 and 1750), and IV.B.1m (vicinity 150 and 2725).

The cyclical nature of the data is somewhat more apparent in the plot of the data averaged over the fifteen years (see Figure IV.B.1p). Additional evidence of this yearly cycle is presented in Figure IV.B.2. This figure presents twelve box plots, one for each month. The data values plotted are the monthly average wind speed for each of the fifteen years. The interquartile range and extreme values are shown in a standard fashion. As an adjunct to this analysis of the year cycle, the coefficient of variation for the monthly averages was computed. The coefficient of variation was essentially constant. See Table IV.B.1. This will have an impact on the choice of the type of model used to model this data.

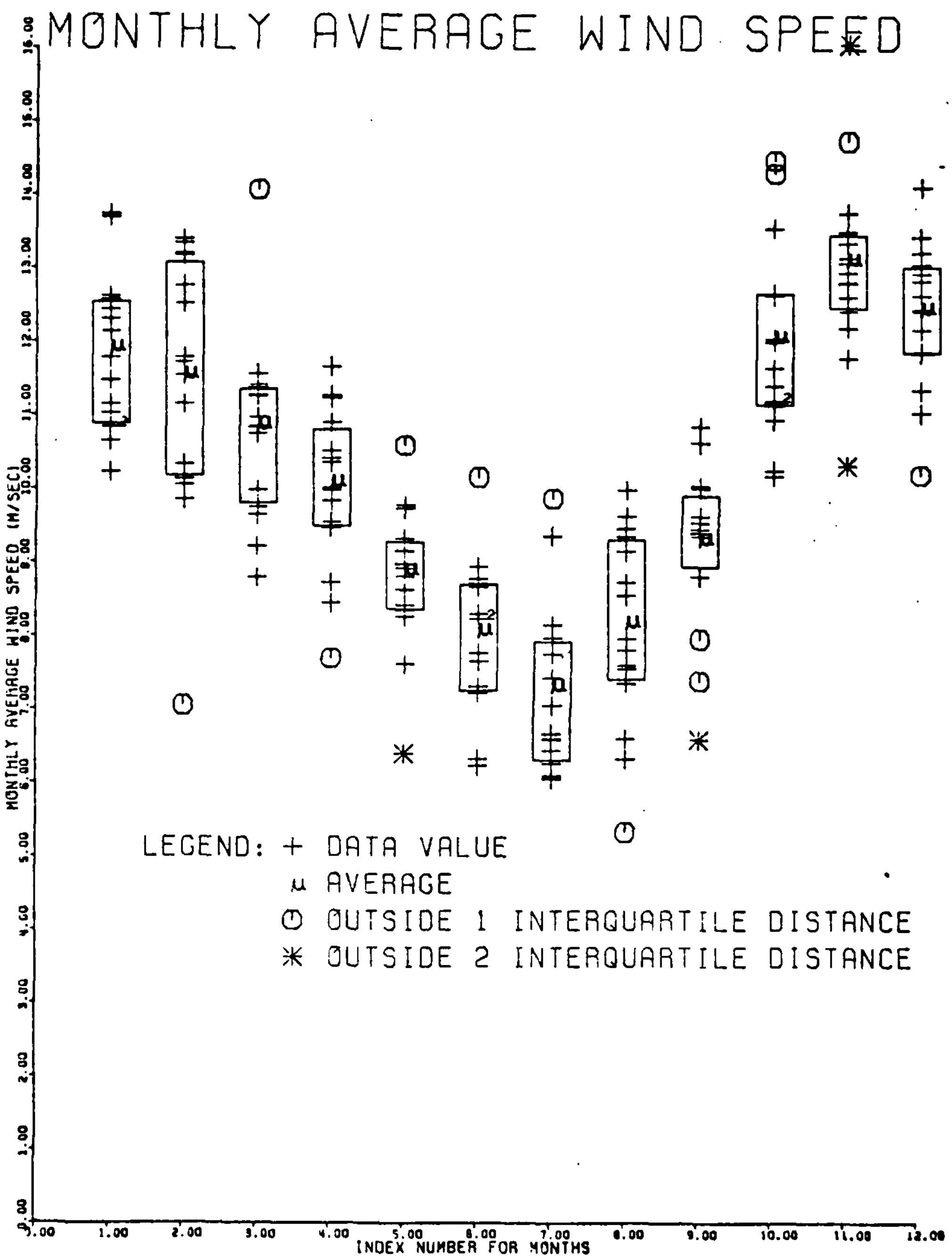


Figure IV.B.2.
Box plot for average wind speed for each month for 15 years.

Table IV.B.1. Average wind velocity by month for each year.

YEAR	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
****	***	***	***	***	***	***	***	***	***	***	***	***
55	11.5	10.3	10.9	10.4	10.6	8.2	6.6	7.6	9.5	14.5	13.5	14.1
56	13.7	13.3	11.4	10.0	9.7	8.8	6.3	7.8	7.4	12.6	12.4	12.4
57	11.2	12.8	10.0	8.4	8.6	8.7	7.1	6.3	9.4	10.2	13.2	13.1
58	11.0	13.2	9.2	9.5	8.4	7.2	8.0	10.0	9.5	12.1	12.8	11.9
59	12.1	11.5	14.1	9.8	8.3	8.9	9.9	9.5	10.0	10.3	12.8	12.8
60	12.3	11.8	11.6	11.7	9.8	8.7	7.4	9.6	9.4	12.0	16.1	12.4
61	12.4	11.7	11.0	10.5	9.3	7.8	6.1	7.6	9.4	12.7	12.6	10.2
62	10.8	7.1	9.6	8.7	7.6	6.2	6.1	7.4	8.0	10.9	14.8	11.4
63	11.8	9.9	11.4	10.0	8.8	8.2	6.4	5.3	9.4	14.3	12.2	11.9
64	13.7	13.2	11.3	11.3	9.2	7.3	7.8	9.1	8.8	11.2	13.4	12.2
65	12.6	11.1	9.8	11.2	9.0	7.6	8.1	6.6	6.6	13.6	11.8	13.2
66	10.2	10.1	11.3	7.7	9.1	7.2	6.1	8.7	10.6	11.2	13.1	11.0
67	12.6	13.4	10.8	9.5	8.4	6.3	6.6	8.0	10.0	11.6	10.3	12.9
68	10.6	10.1	10.8	10.4	6.4	10.2	6.7	9.4	10.9	11.4	13.8	12.6
69	10.8	12.5	8.8	10.9	8.9	8.3	9.4	8.6	9.6	11.1	13.0	13.4
****	***	***	***	***	***	***	***	***	***	***	***	***
AVG	11.84	11.47	10.79	10.00	8.81	7.98	7.23	8.10	9.23	11.98	13.04	12.38
SD	1.04	1.70	1.21	1.07	0.95	1.00	1.16	1.30	1.11	1.29	1.25	0.97
CV	0.09	0.15	0.11	0.11	0.11	0.13	0.16	0.16	0.12	0.11	0.10	0.08

This yearly cycle is also shown in the periodogram and the log of the periodogram of the data (averaged over 15 years) as presented in Figures IV.B.3 and IV.B.4, respectively. The periodogram is computed from the data in the following way. Let $\{X_n,\ n = 1,2,\ldots,N\}$ be the raw data and let $\overline{X} = \sum_{i=1}^{N} X_i$ be the mean of the $\{X_n\}$ sequence and σ_X^2 be the variance. Let the $\{Y_n,\ n = 1,2,\ldots,N\}$ be formed from the $\{X_n\}$ sequence using the relation

$$Y_n = X_n - \overline{X}, \qquad \text{(IV.B.1)}$$

where N = 2920 is an even number. The Fourier transform of the $\{Y_n\}$ sequence will have both a real and complex component and will have $\frac{N}{2}$ elements. Let $\{Z_n,\ n = 1,2,\ldots,\frac{N}{2}\}$ be the Fourier transform of the $\{Y_n\}$ sequence and let Z_{jR} and Z_{jI} be the real and imaginary components of the j^{th} element of $\{Z_n\}$, respectively. Let P_j be the j^{th} element of the periodogram. Then

$$P_j = (Z_{jR}^2 + Z_{jI}^2)/2\pi N\sigma_X^2, \quad j = 1,2,\ldots,\frac{N}{2} \qquad \text{(IV.B.2)}$$

defines the periodogram of the $\{X_n\}$ sequence.

The periodogram dramatically presents the yearly cycle ($j = 1$) as the dominant effect ($P_1 > 150$), although there is some indication of a six month cycle ($j = 2$, $P_2 \approx 9.0$). Somewhat surprising is the apparent lack of any strong time of day effect. The log periodogram reinforces the dominant role of the yearly cycle and indicates that six month and six and twelve hour cycles ($j = 2$, $j = 1460$, $j = 2920$ respectively) may be important.

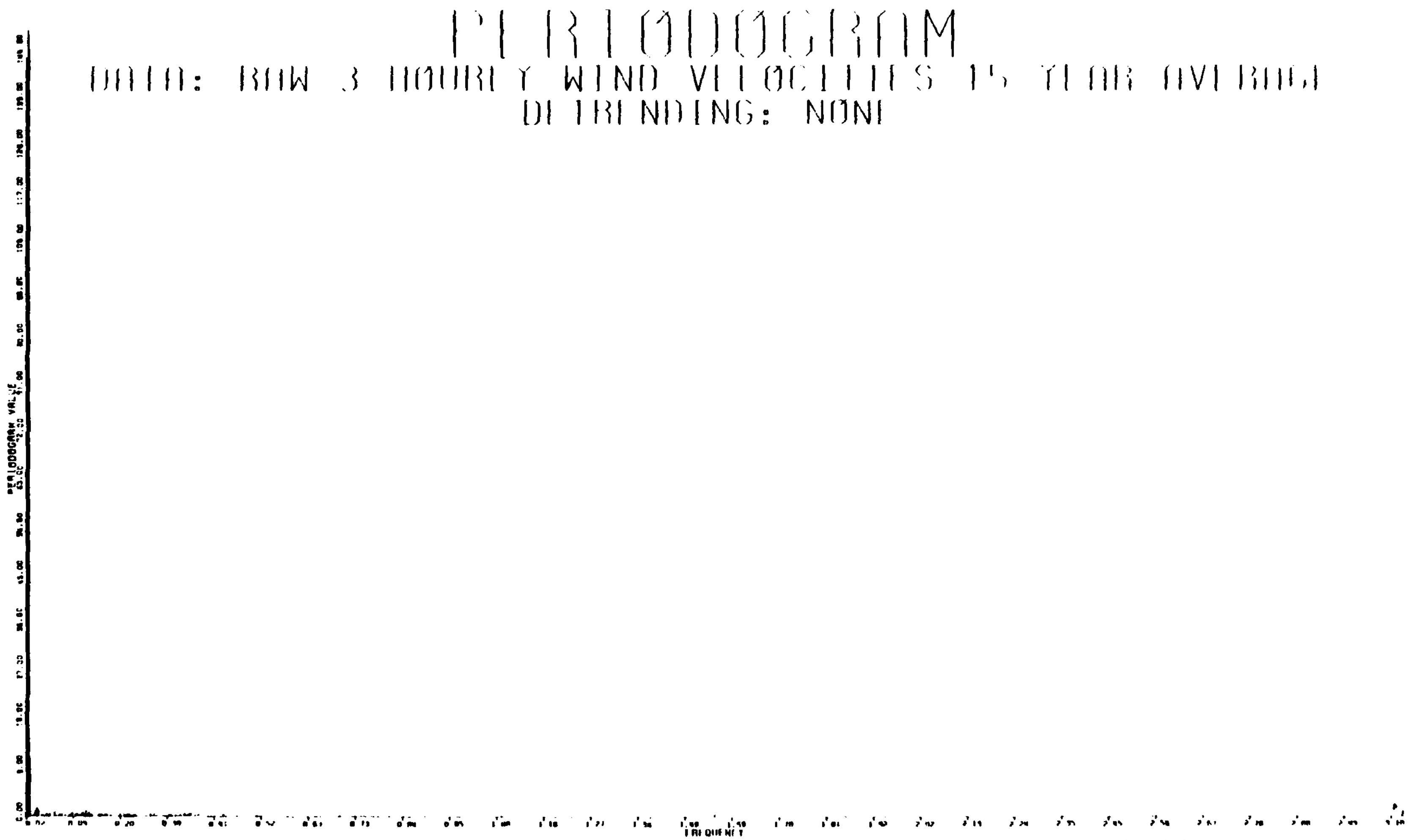


Figure IV.B.3. This periodogram of the averaged raw data shows the dominance of the cycle at one year (w_1) and its second harmonic at w_2 (six months).

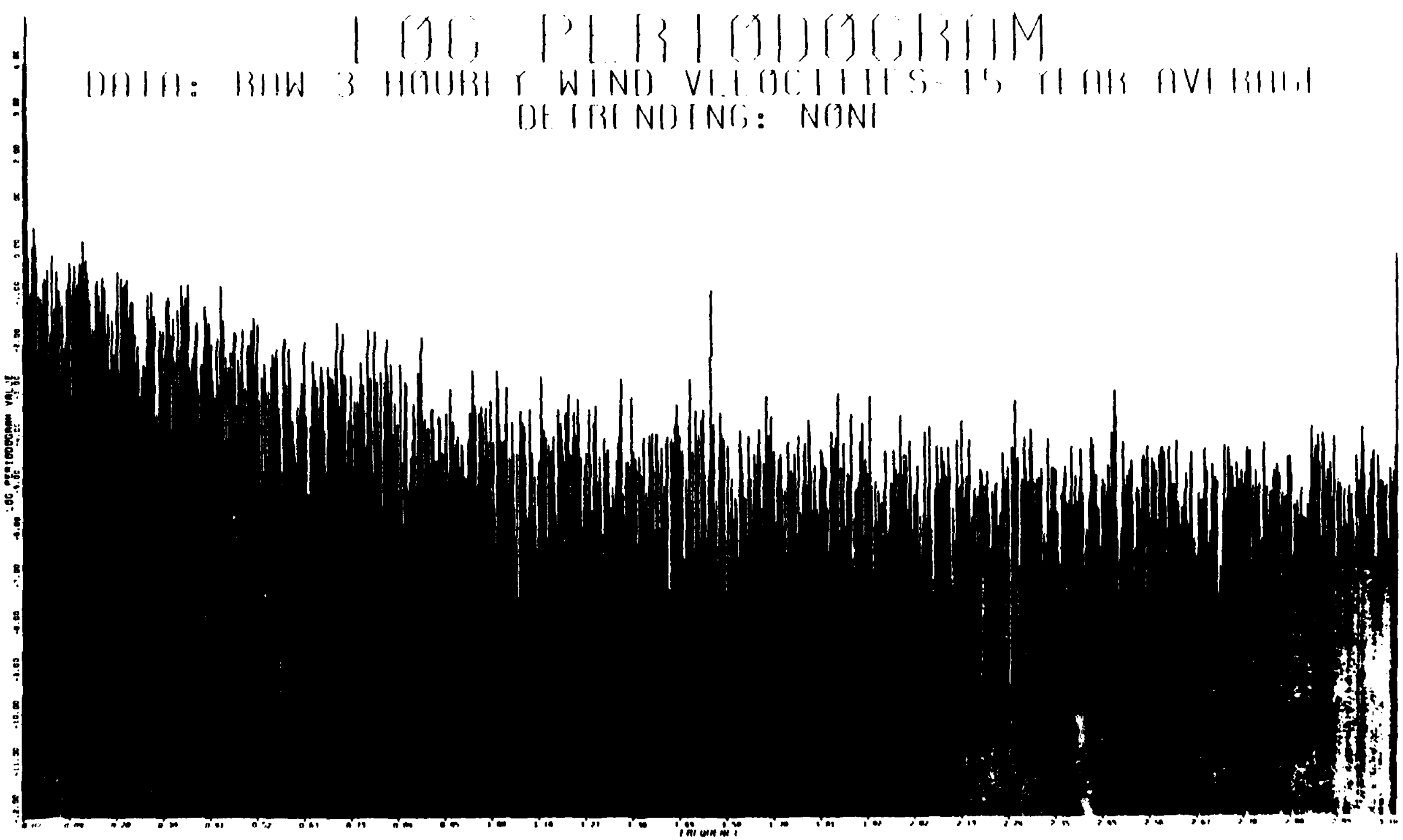


Figure IV.B.4. The log of the periodogram of Figure IV.B.1. shows the harmonics at 6 hours and 12 hours, and a background spectrum closely resembling that of a first order autoregressive process.

The correlation structure of the data is presented in Table IV.B.2. The column indicated as "15 Yr Avg" is the average of the values of the fifteen years. The column indicated as "15 Yr SD" provides the standard deviation of the values about their mean. It is not the standard deviation of the average. This latter quantity can be obtained by dividing by the square root of 15. The last column provides the correlation structure of the average data. The estimated correlations remain artificially high in this case because averaging reduces the variability of the data about the year cycle which intensifies the artificial increase in correlation due to the year cycle. The correlation structure revealed in Table IV.B.2 for individual years closely resembles that of an AR(1) model, in that the k-step correlation is approximately the one-step correlation raised to the k^{th} power. The correlations in the table have a tendency to be slightly higher than the theoretical, calculated value, but the agreement is reasonably good for about ten steps. Beyond that point the correlations are kept up by the year cycle, which is not as prominent in the yearly data as it is in the averaged data. If nothing else the disparity between the two correlations is evidence of the existence of a trend in the data.

At this point sufficient information is available to determine some characteristics of the general form of the model for representing the wind speed data. As noted above, the correlation structure is similar to that of an AR(1) process

Table IV.B.2. The estimated correlations for the averaged data hold up, clearly showing a long-term trend in the data (in fact the one-year cycle).

DATA: 3-HOURLY WIND VELOCITIES; DETRENDING: NONE

LAG	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	15 YR AVG	15 YR SD	AVG DATA
1	0.82	0.80	0.81	0.81	0.83	0.80	0.81	0.82	0.83	0.80	0.84	0.83	0.82	0.87	0.84	0.823	0.019	0.936
2	0.68	0.66	0.67	0.68	0.70	0.66	0.67	0.70	0.70	0.66	0.71	0.70	0.69	0.74	0.70	0.688	0.022	0.891
3	0.55	0.54	0.54	0.57	0.58	0.55	0.54	0.58	0.58	0.54	0.59	0.56	0.56	0.62	0.56	0.564	0.023	0.847
4	0.44	0.44	0.45	0.48	0.48	0.45	0.44	0.49	0.49	0.44	0.48	0.48	0.47	0.51	0.45	0.467	0.023	0.819
5	0.35	0.36	0.36	0.40	0.39	0.36	0.34	0.41	0.40	0.35	0.39	0.39	0.39	0.42	0.35	0.375	0.026	0.786
6	0.27	0.30	0.30	0.35	0.32	0.30	0.26	0.35	0.33	0.27	0.31	0.34	0.34	0.34	0.28	0.311	0.030	0.768
7	0.21	0.24	0.25	0.32	0.27	0.26	0.19	0.30	0.29	0.21	0.26	0.28	0.29	0.28	0.23	0.258	0.036	0.751
8	0.16	0.21	0.24	0.29	0.24	0.22	0.14	0.28	0.26	0.18	0.22	0.24	0.25	0.24	0.20	0.226	0.040	0.746
9	0.13	0.17	0.19	0.26	0.21	0.19	0.10	0.23	0.23	0.15	0.18	0.19	0.20	0.21	0.17	0.186	0.040	0.729
10	0.11	0.15	0.18	0.22	0.18	0.15	0.06	0.20	0.20	0.11	0.17	0.16	0.18	0.19	0.17	0.161	0.039	0.723
11	0.11	0.15	0.16	0.20	0.15	0.12	0.05	0.17	0.18	0.08	0.15	0.13	0.13	0.17	0.17	0.141	0.039	0.712
12	0.12	0.15	0.16	0.19	0.15	0.10	0.05	0.17	0.17	0.07	0.13	0.11	0.12	0.17	0.17	0.136	0.040	0.714
13	0.12	0.14	0.15	0.19	0.14	0.07	0.04	0.17	0.16	0.06	0.11	0.08	0.09	0.16	0.17	0.122	0.044	0.706
14	0.11	0.15	0.15	0.19	0.13	0.06	0.04	0.17	0.14	0.05	0.11	0.07	0.08	0.15	0.16	0.117	0.046	0.706
15	0.11	0.14	0.14	0.18	0.13	0.05	0.04	0.17	0.14	0.05	0.10	0.05	0.06	0.15	0.15	0.110	0.048	0.703
16	0.11	0.16	0.14	0.18	0.13	0.05	0.04	0.17	0.13	0.06	0.10	0.04	0.06	0.15	0.15	0.112	0.047	0.707
17	0.09	0.14	0.13	0.16	0.12	0.03	0.03	0.15	0.12	0.05	0.10	0.02	0.05	0.16	0.13	0.098	0.049	0.696
18	0.07	0.15	0.13	0.15	0.12	0.02	0.02	0.14	0.11	0.04	0.11	0.01	0.05	0.16	0.12	0.093	0.049	0.689
19	0.06	0.14	0.12	0.14	0.11	0.01	0.02	0.14	0.10	0.04	0.11	-.00	0.04	0.16	0.11	0.085	0.051	0.679
20	0.05	0.13	0.13	0.13	0.11	0.01	0.02	0.14	0.10	0.04	0.12	0.00	0.05	0.15	0.08	0.084	0.049	0.681
21	0.04	0.12	0.12	0.12	0.09	-.00	0.01	0.12	0.10	0.02	0.12	-.00	0.05	0.15	0.06	0.075	0.050	0.670
22	0.04	0.12	0.13	0.12	0.09	0.00	0.00	0.10	0.11	0.03	0.12	0.01	0.07	0.14	0.04	0.075	0.048	0.671
23	0.05	0.12	0.11	0.11	0.09	0.01	-.01	0.08	0.12	0.03	0.12	0.00	0.08	0.13	0.04	0.072	0.046	0.669
24	0.05	0.12	0.12	0.11	0.09	0.01	0.00	0.09	0.13	0.04	0.13	0.01	0.10	0.11	0.02	0.075	0.045	0.677
25	0.05	0.11	0.10	0.10	0.08	0.01	-.00	0.06	0.12	0.02	0.13	0.01	0.08	0.10	0.00	0.064	0.045	0.673
26	0.04	0.12	0.09	0.09	0.08	0.01	-.00	0.05	0.11	0.02	0.13	0.02	0.07	0.08	-.01	0.059	0.043	0.672
27	0.03	0.10	0.07	0.09	0.07	0.02	-.01	0.05	0.10	0.01	0.12	0.00	0.05	0.07	-.01	0.051	0.041	0.671
28	0.03	0.09	0.07	0.08	0.08	0.03	0.00	0.05	0.09	0.01	0.11	0.01	0.04	0.07	-.02	0.051	0.038	0.675
29	0.03	0.08	0.06	0.07	0.08	0.03	0.00	0.04	0.07	0.01	0.10	0.01	0.03	0.07	-.01	0.044	0.033	0.677
30	0.03	0.08	0.06	0.06	0.08	0.04	-.00	0.04	0.05	0.00	0.10	0.02	0.03	0.08	-.01	0.045	0.032	0.683
31	0.04	0.08	0.05	0.06	0.09	0.03	0.0	0.03	0.05	-.00	0.09	0.02	0.02	0.10	-.00	0.044	0.034	0.681
32	0.05	0.09	0.05	0.07	0.10	0.03	0.02	0.05	0.05	0.00	0.08	0.03	0.03	0.13	0.01	0.053	0.034	0.689

with positive correlation although nuances may appear as the year cycle is removed. Hence, an AR(1) model should be used as a starting point for the construction of the model for wind speed.

In addition, the cyclic nature of the process must be modeled. This can be done with either an additive or multiplicative model. An additive model might have a structure as follows.

$$X_n = \mu_n + \varepsilon_n, \qquad \text{(IV.B.3)}$$

where $\{X_n\}$ is the time series under consideration, μ_n is a deterministic function of n, and the innovative sequence $\{\varepsilon_n\}$ is a stationary sequence of random variables. In the usual model this stationarity implies that the marginal variance σ^2 is constant and the correlations only depend on the lag (i.e., $\rho(X_n, X_{n+k}) = \rho(k)$). Using the same definitions the multiplicative model would have the form

$$X_n = \mu_n \varepsilon_n, \qquad \text{(IV.B.4)}$$

where again the $\{\varepsilon_n\}$ sequence is stationary and independent of μ_n. A characteristic of the additive model is that the coefficient of variation is a function of the value of μ_n. The multiplicative model produces a constant coefficient of variation. Since the data has a coefficient of variation that

is essentially constant, in the crude monthly analysis, the multiplicative model is preferred.

We have yet to determine the exact form of the mean, μ_n, in equation IV.B.4. However, we do know that this mean will have a yearly cyclic nature. We also have yet to determine the general structural nature of the innovative process ε_n. These subjects are addressed in the following sections.

C. THE FORM OF THE MEAN; DETRENDING THE DATA

Two basic models were considered to represent the mean. The first was a single harmonic sinusoidal model

$$\mu_n = a + b_1 \sin\left(\frac{2\pi n}{2920}\right) + b_2 \cos\left(\frac{2\pi n}{2920}\right) = a + k \cos\left(\frac{2\pi n}{2920} + \theta\right), \tag{IV.C.1}$$

where $k = (b_1^2 + b_2^2)^{1/2}$ and $\theta = \tan^{-1}(-\frac{b_1}{b_2})$. The second was an exponential sine with one harmonic

$$\mu_n = e^{a+b_1\sin\left(\frac{2\pi n}{2920}\right)+b_2\cos\left(\frac{2\pi n}{2920}\right)} = e^a e^{k\cos\left(\frac{2\pi n}{2920}+\theta\right)} \tag{IV.C.2}$$

The second model has the theoretical advantage that it can not be negative and will represent higher harmonics in a compact form. The sinusoidal model may or may not be negative depending on the values for a, b_1, and b_2. In spite of the theoretical preference for the exponential sine, both models were used initially to see if either produced significantly

better results. Note that if k is small the models are hardly distinguishable. If k is large, the exponential sin will clip at low values and is a cycle that would have many harmonics in its Fourier transform.

The values for the constants in equation IV.C.1 were determined in a straightforward procedure using the least-squares regression procedure of MINITAB and the data averaged over 15 years. These estimates could also have been obtained from the periodogram at $\omega_1 = 2\pi N$, $I(\omega_1) \sim \{(\hat{b}_1)^2 + (\hat{b}_2)^2\}$, using the relations

$$\hat{a} = \sum_{i=1}^{N} \frac{X_i}{N} = \bar{X}; \qquad \text{(IV.C.3)}$$

$$\hat{b}_1 = 2 \sum_{i=1}^{N} X_i \sin(\frac{2\pi i}{N})/N = \text{imaginary component of periodogram at } 2\pi/N; \qquad \text{(IV.C.4)}$$

$$\hat{b}_2 = 2 \sum_{i=1}^{N} X_i \cos(\frac{2\pi i}{N})/N = \text{real component of periodogram at } 2\pi/N.$$

The variance of these estimates is $2\sigma_\varepsilon^2/N$ if the X_i's are independent, but since this is clearly not the case here, estimates of the variance of the estimates cannot be obtained directly. The results of the estimation are contained in column 1 of Table IV.C.1.

Similar results were obtained for the constants in IV.C.2 by a slightly more complicated procedure. In order to use a

TABLE IV.C.1

Parameter Estimates for Models of the Mean Value Function of the Wind Velocity Data

PARAMETER	ESTIMATE 1 harmonic sine	1 harmonic exp sine	2 harmonic sine	2 harmonic exp sine	harmonic exp sine
a/a'	10.230	2.309	10.230	2.307	2.307
b_1	-0.176	-0.011	-0.175	-0.011	-0.011
b_2	2.560	0.260	2.566	0.260	0.260
b_3	-	-	-0.593	-0.057	-0.057
b_4	-	-	-0.397	-0.054	-0.054
b_5	-	-	-	-	0.014
b_6	-	-	-	-	0.001
b_7	-	-	-	-	-0.010

least squares approach, a linear relationship must be established for the mean value of the process. Taking logs is the obvious technique to employ, but this introduces a complication. Taking logs and expectation of IV.C.2 we have

$$E(\ln X_n) = \ln \mu_n + E(\ln \varepsilon_n)$$

$$= a + b_1 \sin(\tfrac{2\pi n}{2920}) + b_2 \cos(\tfrac{2\pi n}{2920}) + c.$$

For example, if the $\{\varepsilon_n\}$ sequence is marginally distributed as a unit Gamma variate, $G(1,k)$, then $c = \psi(k)$, where $\psi(k)$ is the digamma function (derivative of $\ln \Gamma(k)$). See Cox and Lewis [Ref. 29], pages 24-27. The value of the constant c will be combined with the constant a in the least squares estimation using the $\ln X_n$'s, giving the constant $a' = a+c$. To estimate a+c without making Gamma assumptions for the innovative process, the X_n's are divided by

$$\mu_n' = \hat{b}_1 \sin(\tfrac{2\pi n}{2920}) + \hat{b}_2 \cos(\tfrac{2\pi n}{2920})$$

to give X_n'. The data is then divided by the average of the X_n's which estimate $e^{-(a+c)}$. The result of this is a series with mean value (within statistical fluctuation) of 1 if the model for the cycle is correct. The values obtained are listed in column 2 of TAble IV.C.1. The results of these estimates are in Figure IV.C.1. In this figure the average data

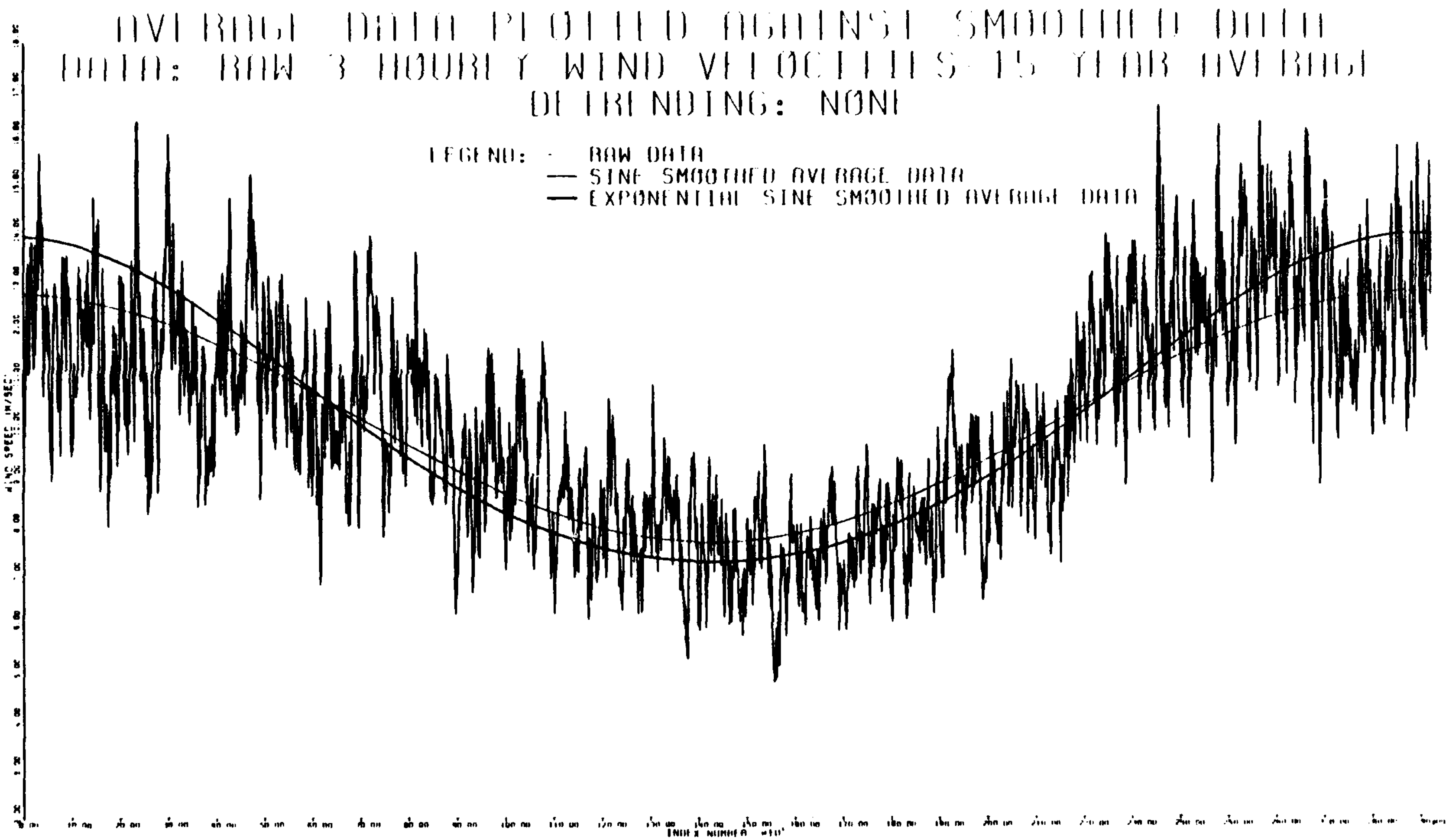


Figure IV.C.1. The disparity between the two-year trend functions and the data indicates the need for a second (6-month) harmonic in the function. This is also indicated in the periodogram (Fig. IV.B.1.).

is plotted against the value computed for μ_n using both of the models under consideration.

When using a multiplicative model, the residuals are formed by dividing the raw data by the mean. The results of this procedure are presented in Figures IV.C.2a through IV.C.2p using the exponential sine model for the mean. The results are not significantly different using the sinusoidal model of the means. Hence, only the results for the average data are presented for this case in Figure IV.C.3.

The log periodogram of the average data detrended using the sinusoidal model for the mean is shown in Figure IV.C.4. A five-step moving average of this log periodogram is presented in Figure IV.C.5. The detrending has clearly reduced the importance of the yearly cycle, but still shows some evidence of a six month cycle and six and twelve hour cycles. Similar information is provided for the average data detrended using the exponential sine model for the mean in Figures IV.C.6 and IV.C.7. This model does not reduce the effect of the yearly cycle as much as the sinusoidal model for the mean, but still shows the six month cycle as being important and some evidence of six and twelve hour cycles.

Since the exponential sine has the theoretical advantage of being non-negative and both models of the mean produce similar results when applied to the data, the exponential sine is selected as the model of choice and the analysis is continued using it exclusively.

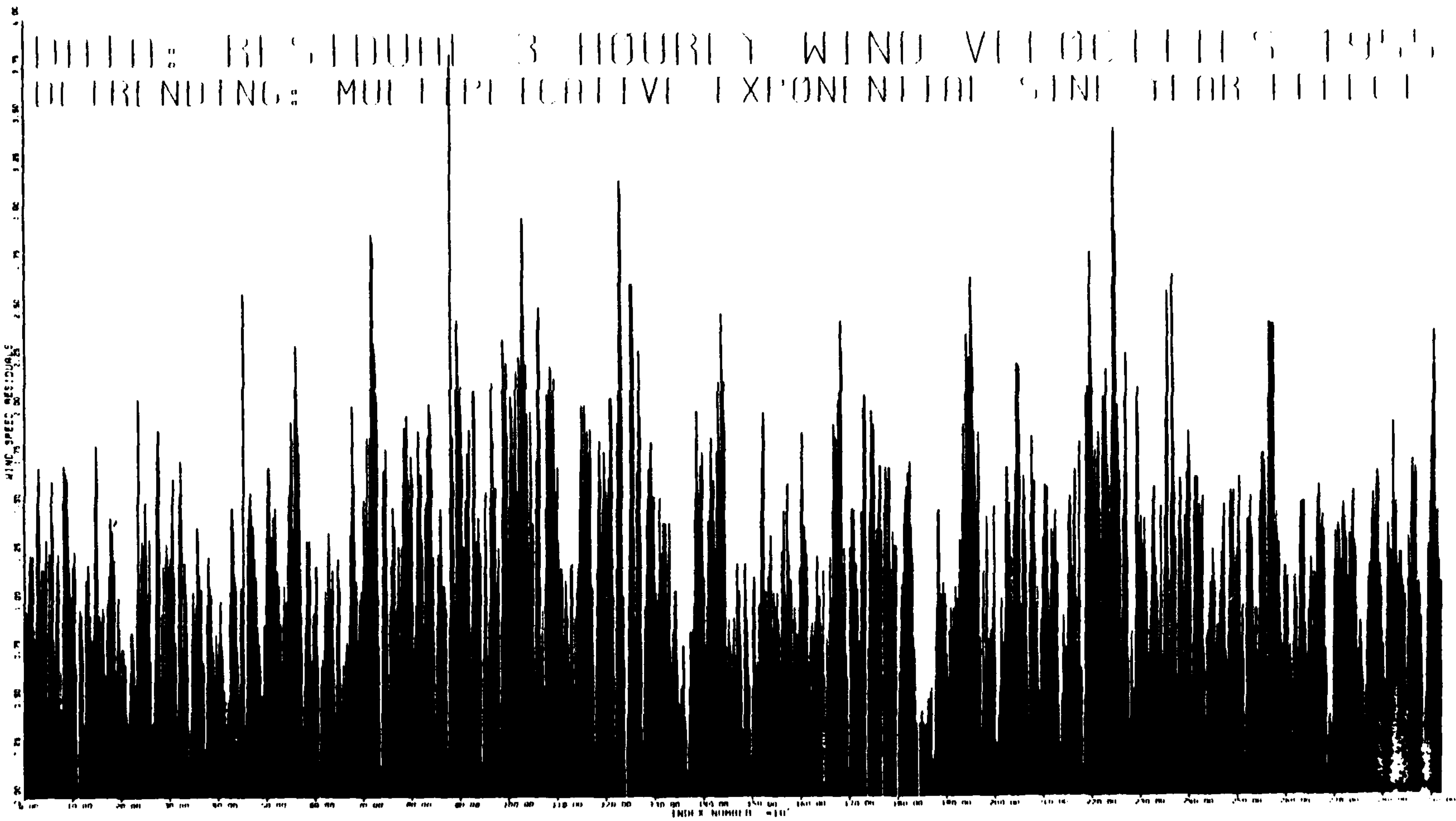


Figure IV.C.2a. 1955 Detrended data.

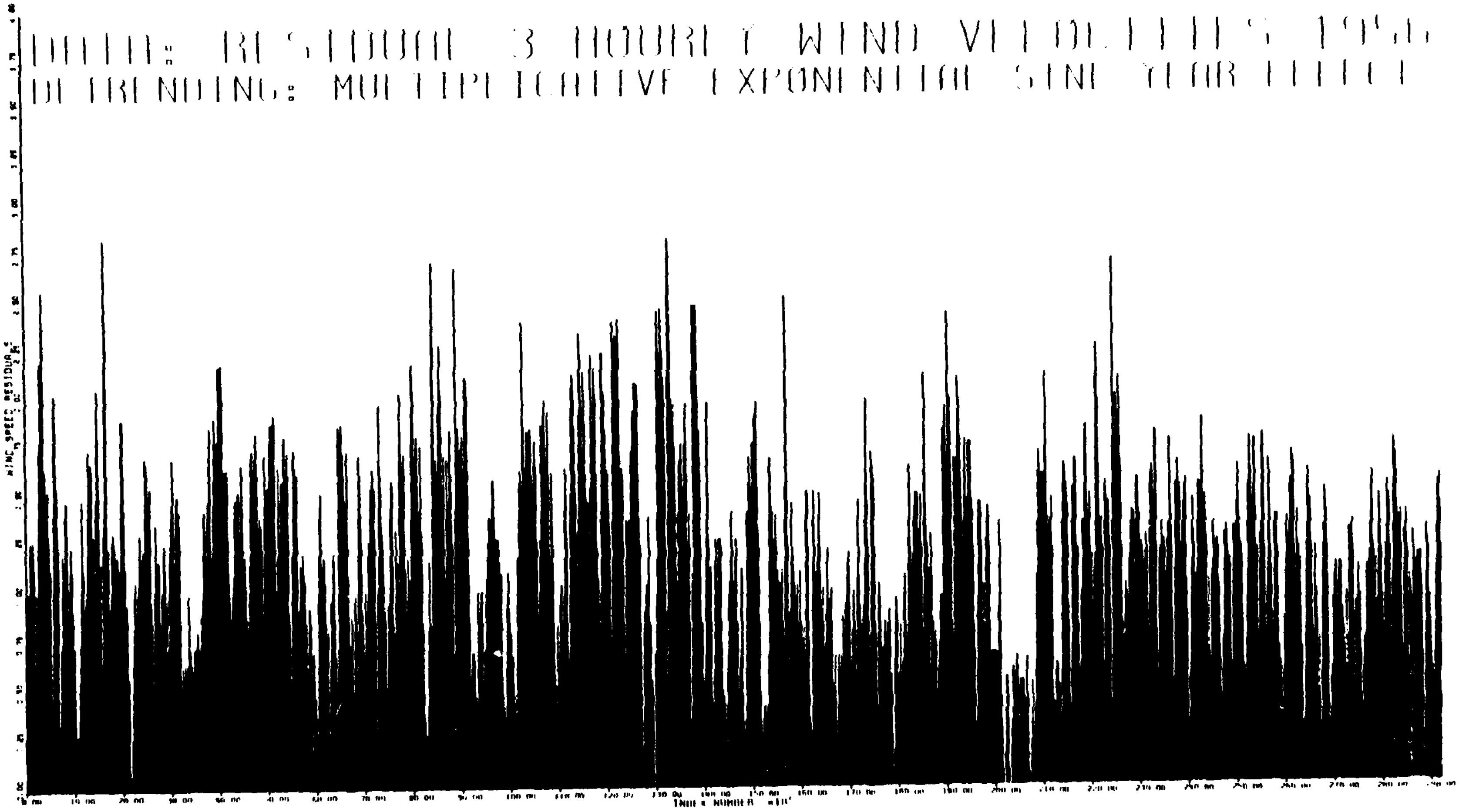


Figure IV.C.2b.   1956 Detrended data.

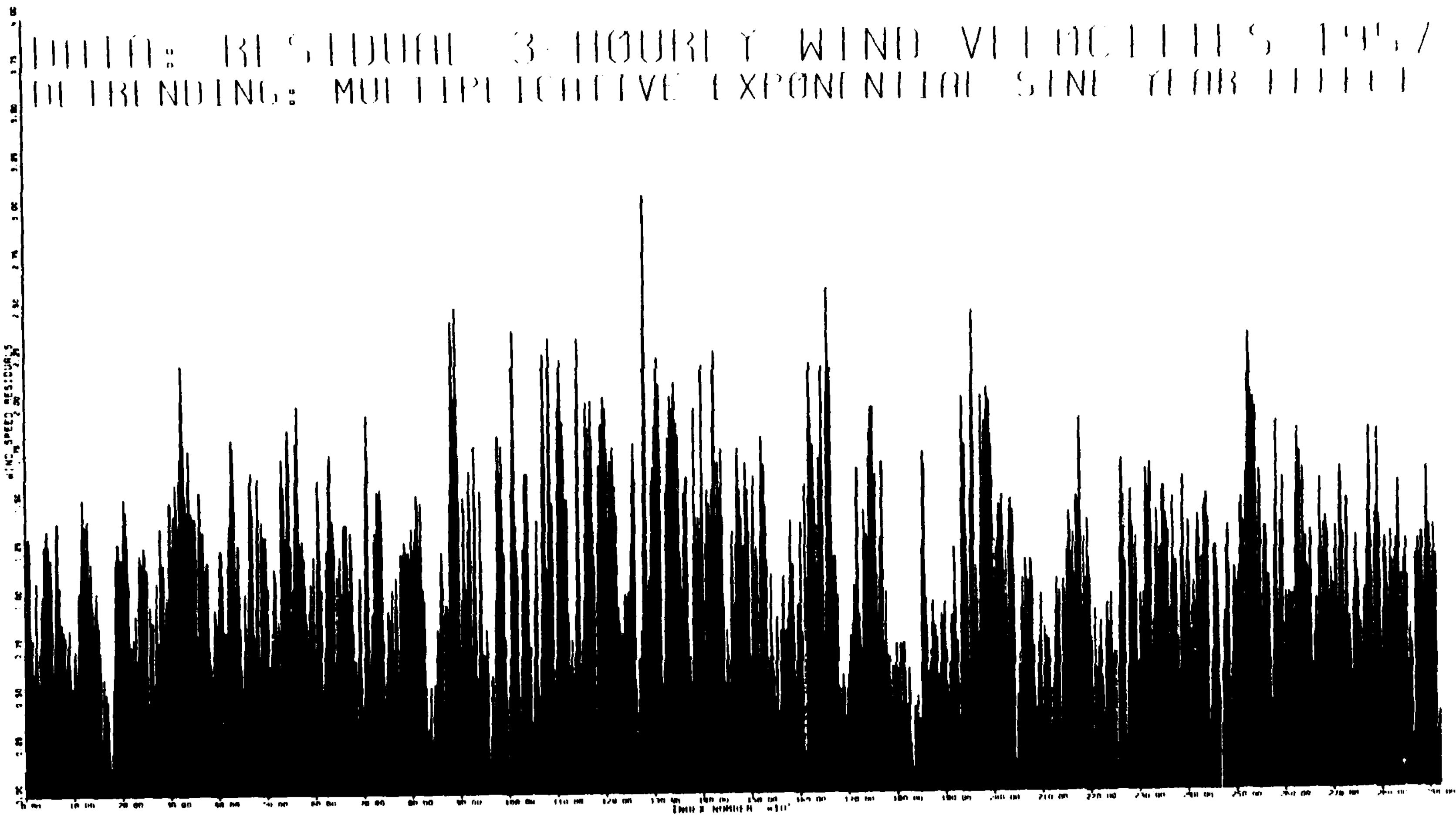


Figure IV.C.2c. 1957 Detrended data.

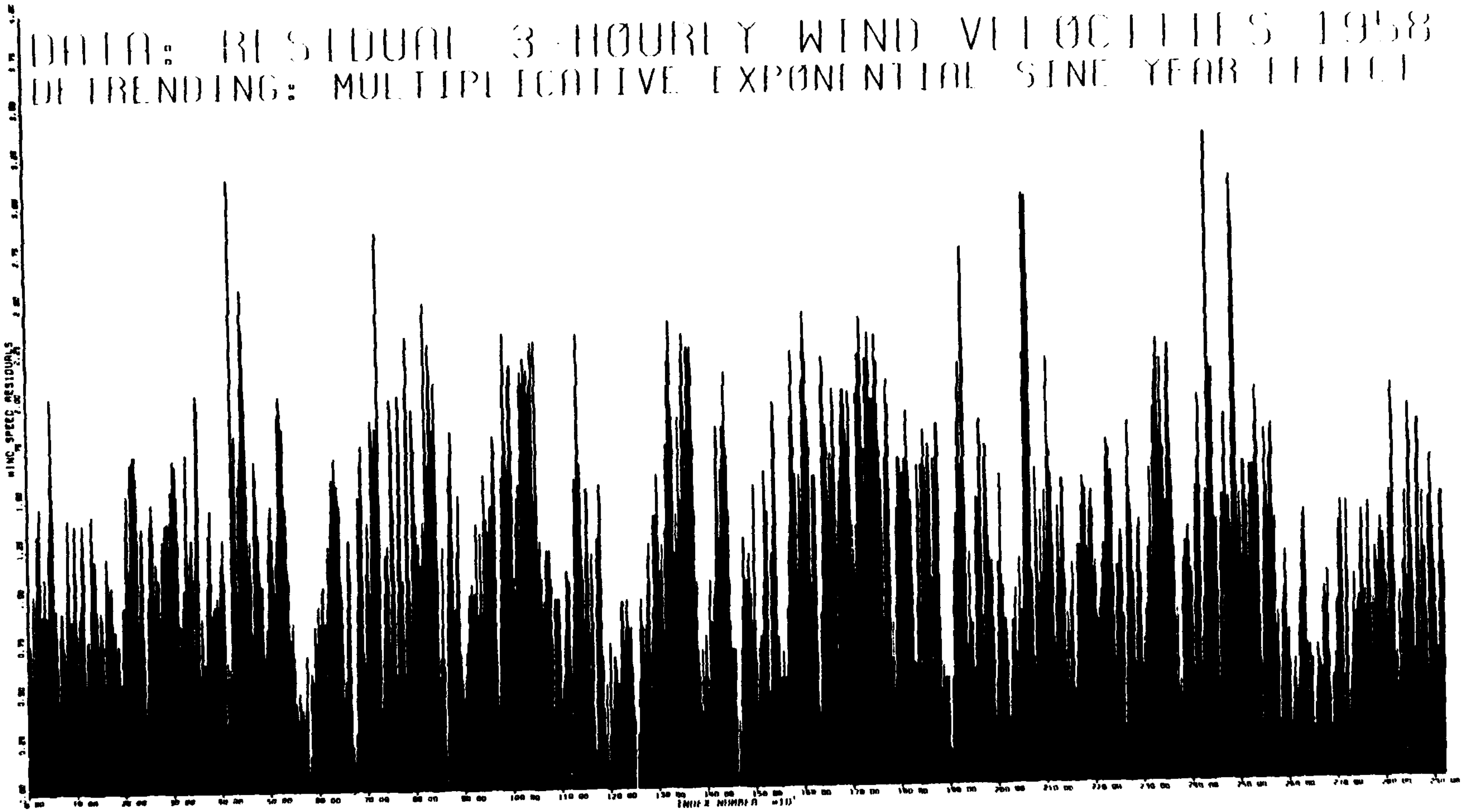


Figure IV.C.2d. 1958 Detrended data.

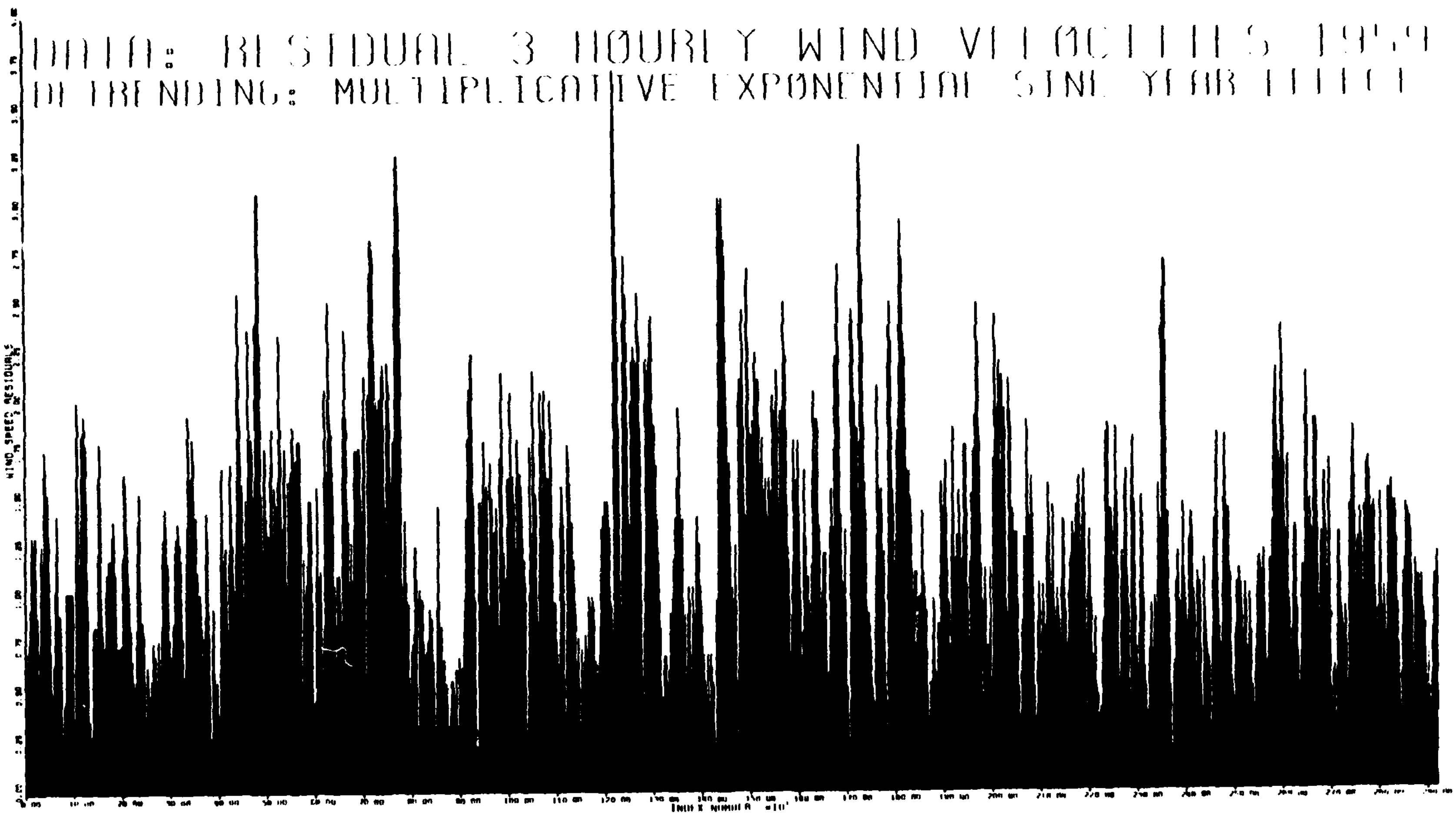


Figure IV.C.2e. 1959 Detrended data.

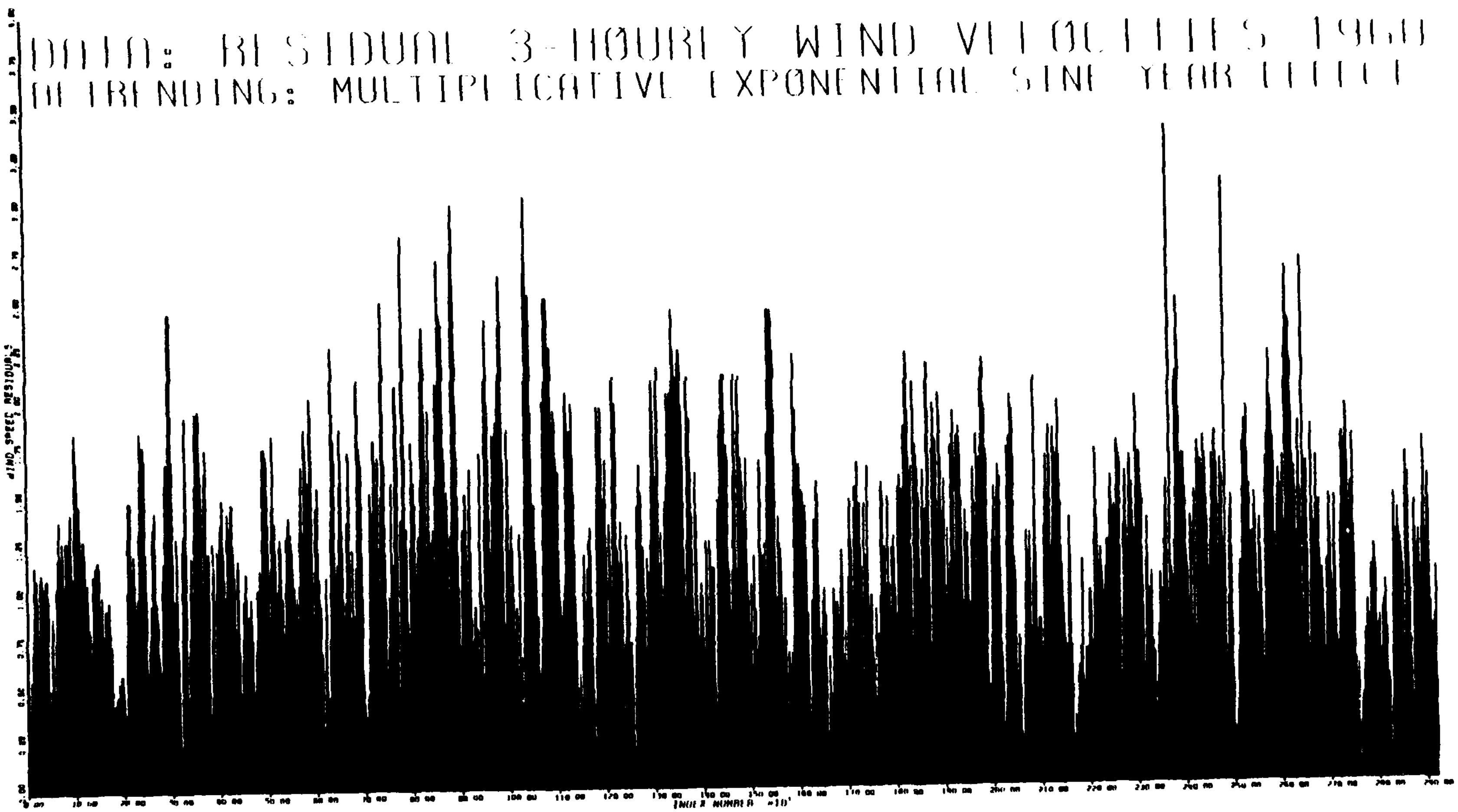


Figure IV.C.2f. 1960 Detrended data.

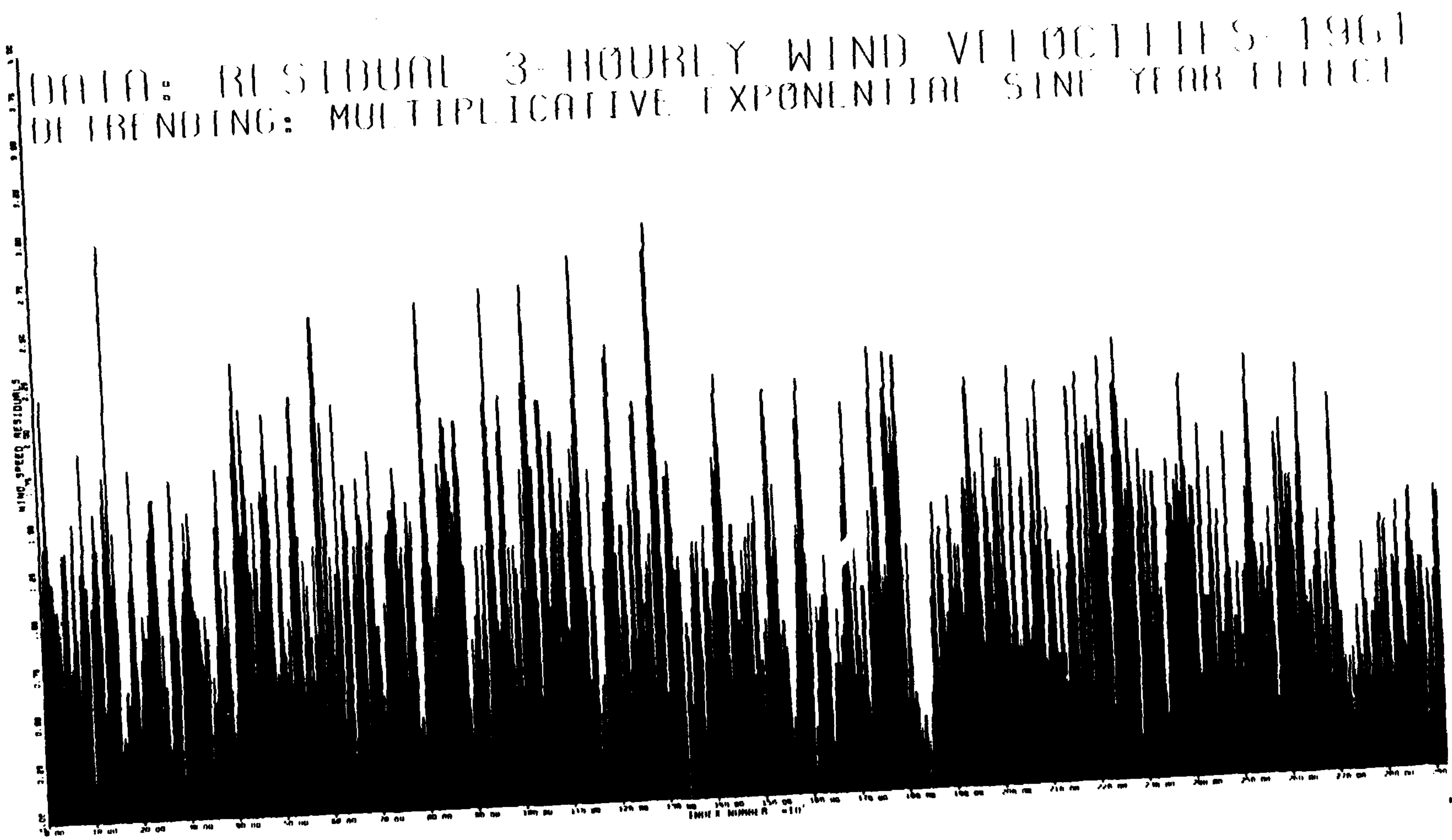


Figure IV.C.29.   1961 Detrended data.

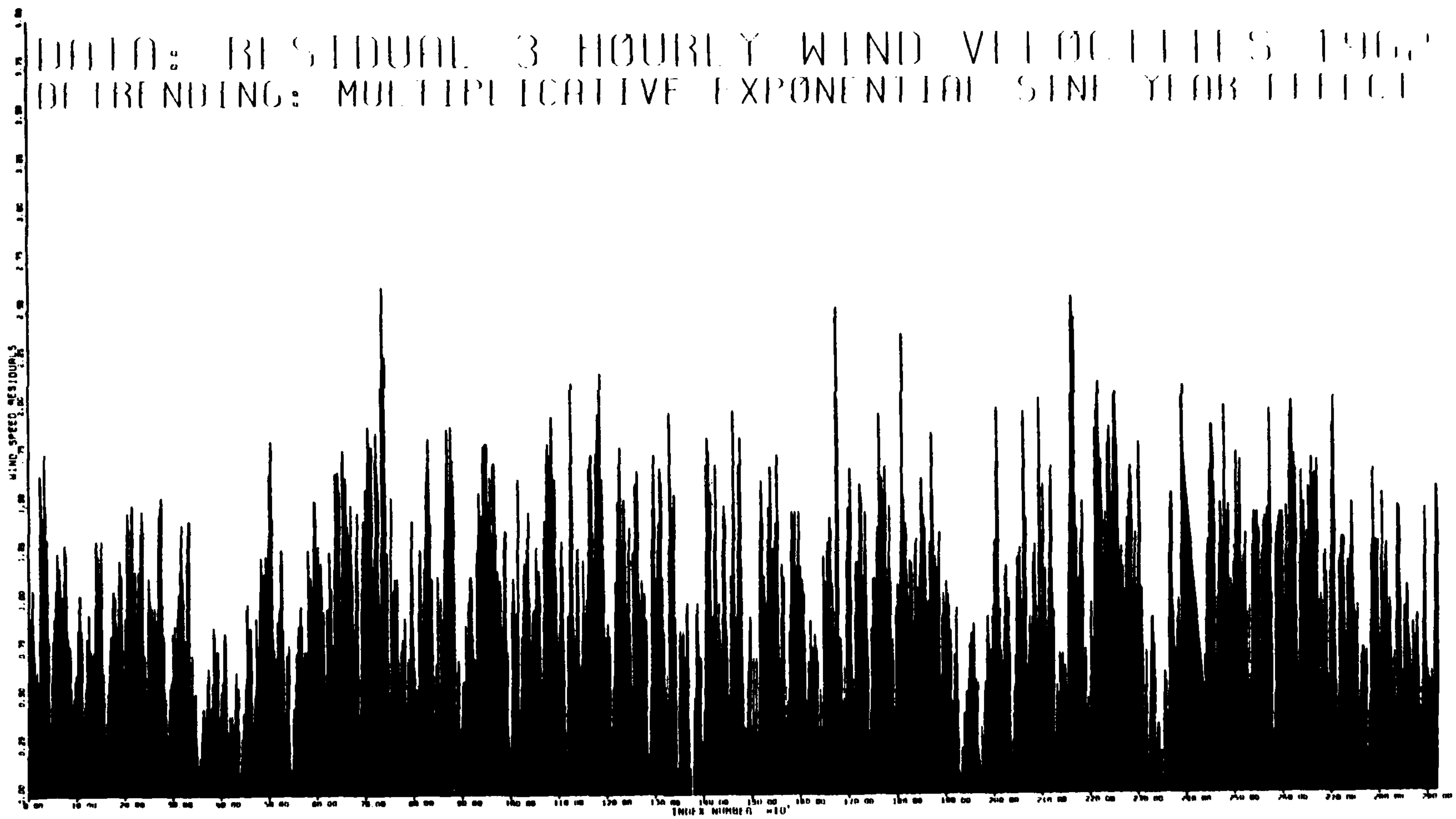


Figure IV.C.2h. 1962 Detrended data.

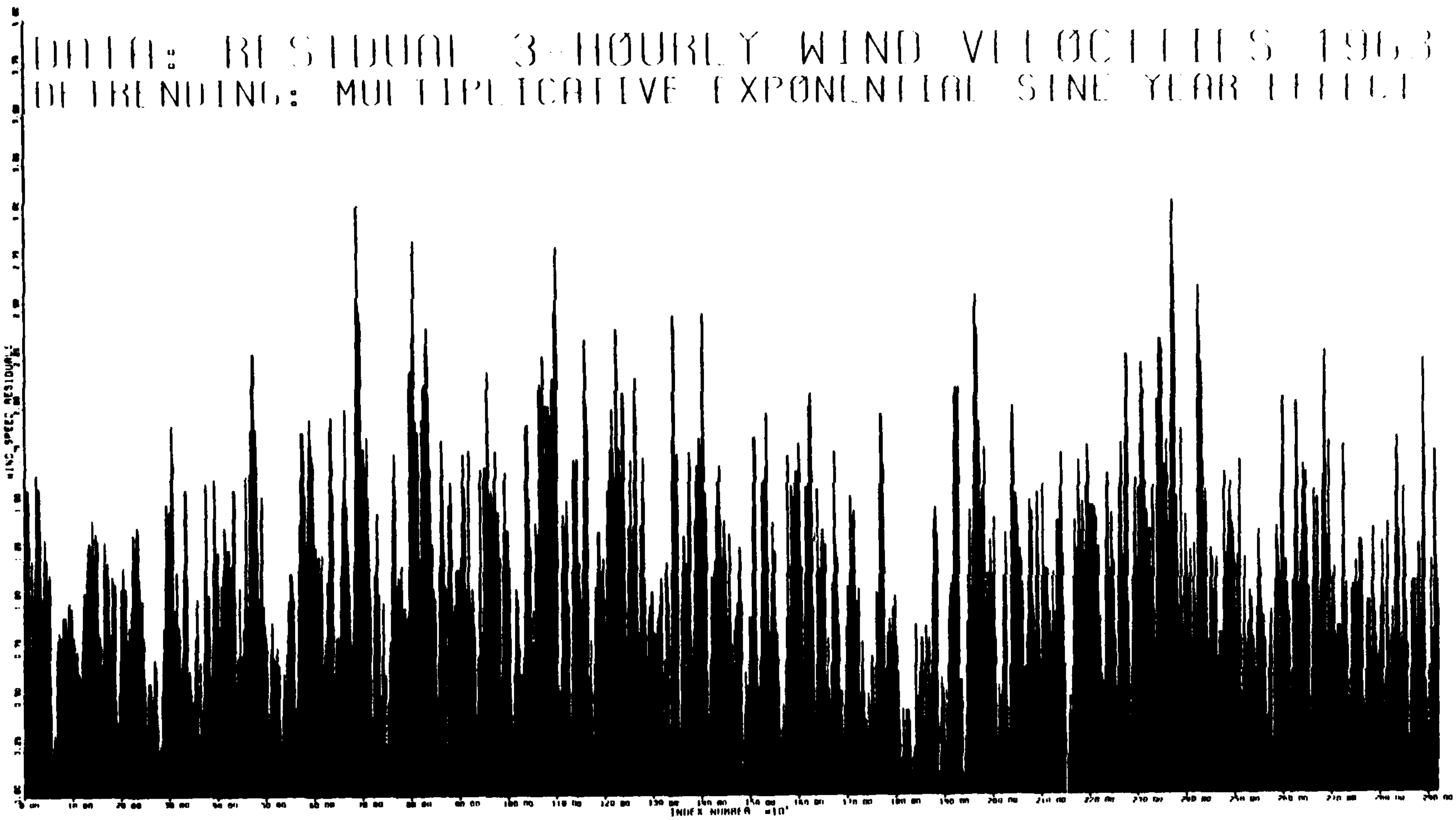


Figure IV.C.2i. 1963 Detrended data.

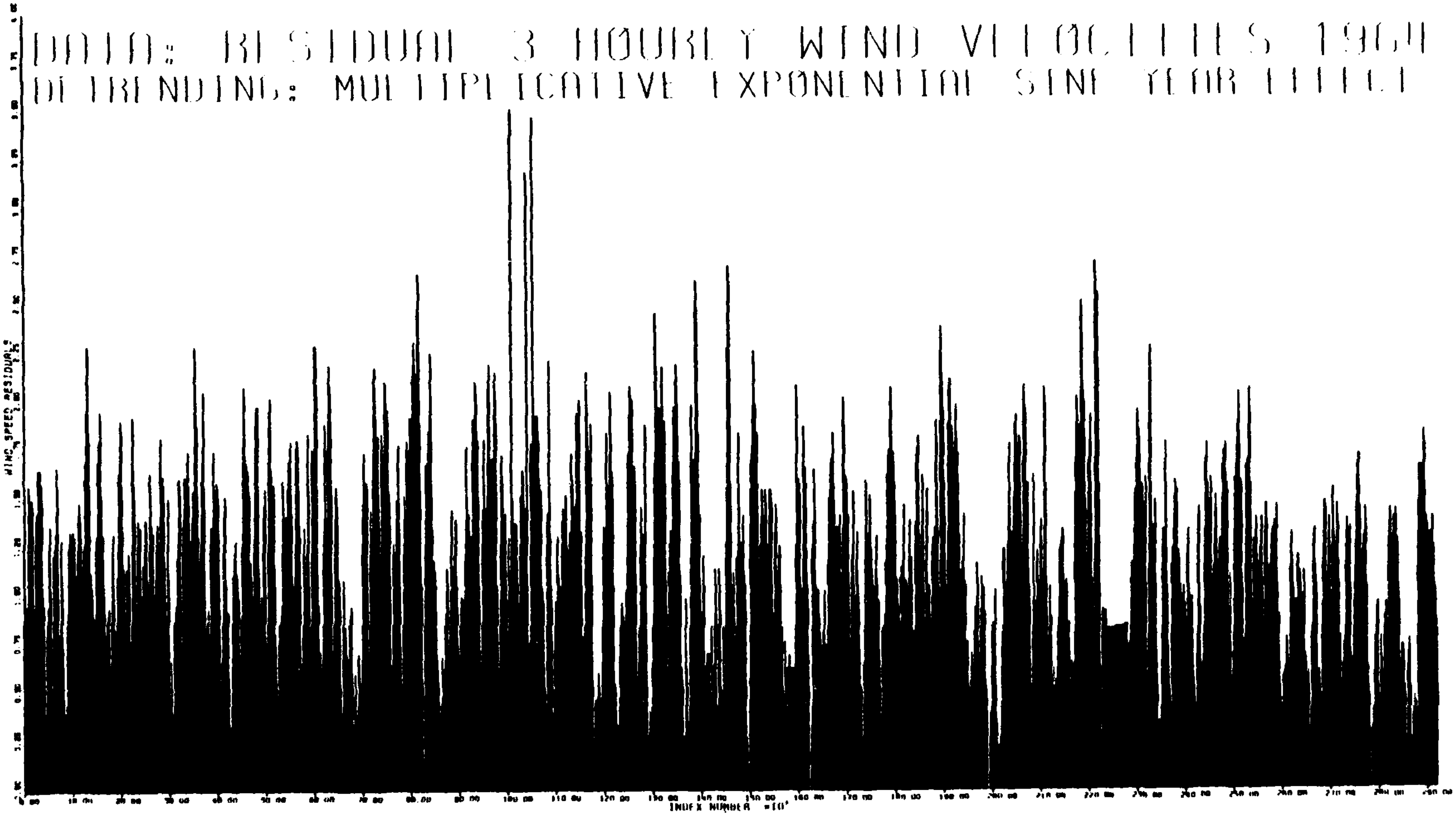


Figure IV.C.2j. 1964 Detrended data.

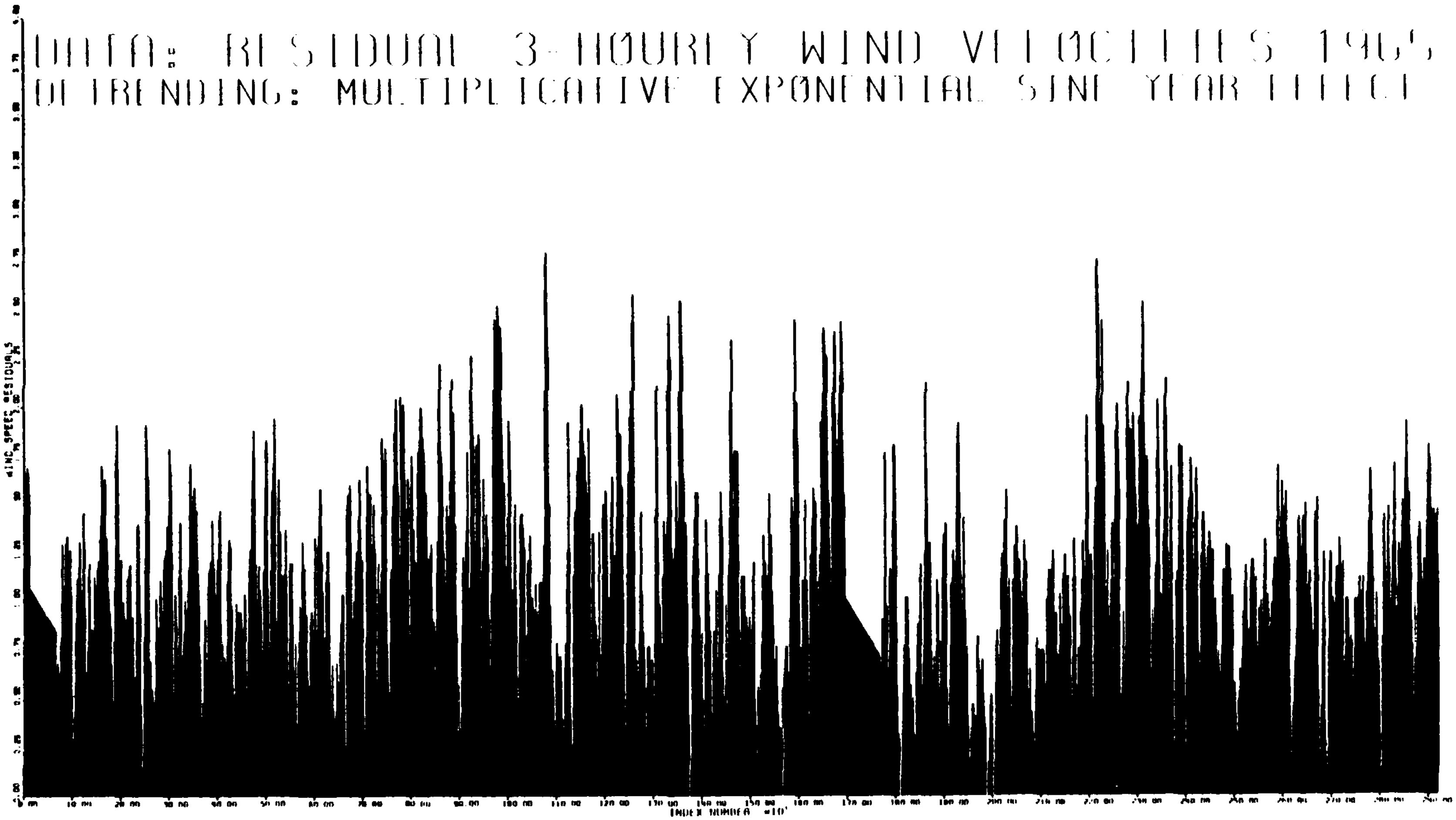


Figure IV.C.2k. 1965 Detrended data.

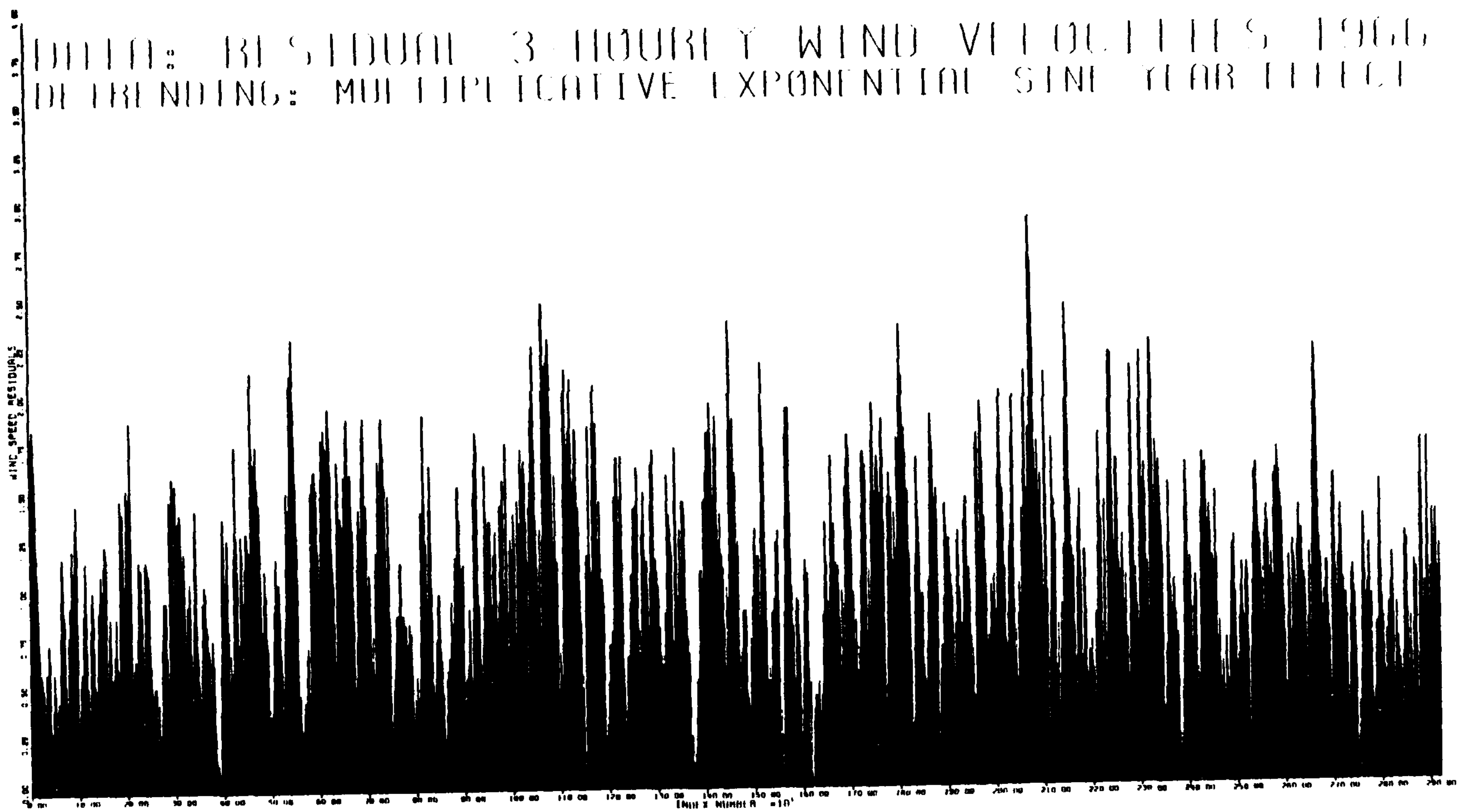


Figure IV.C.21. 1966 Detrended data.

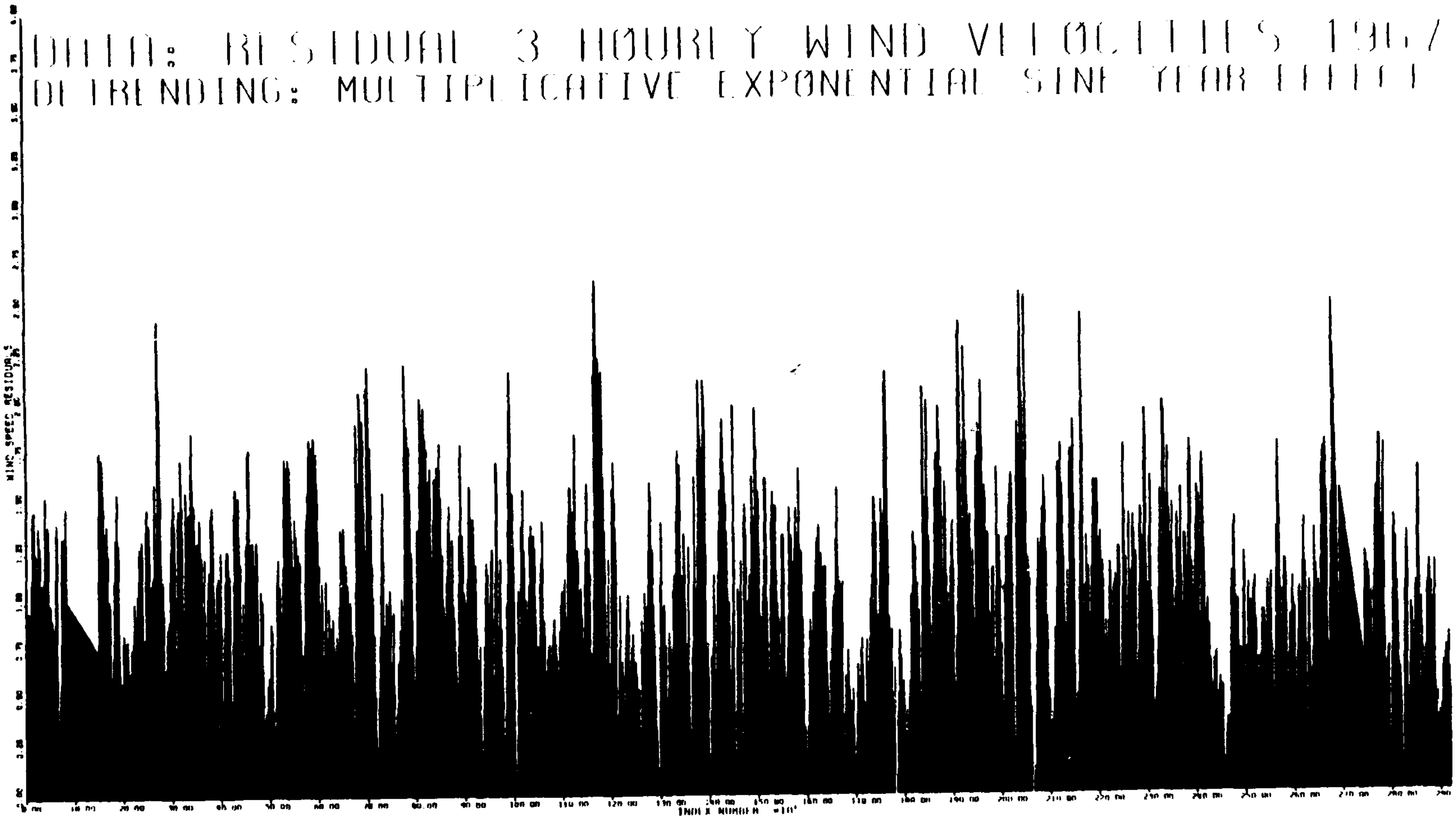


Figure IV.C.2m. 1967 Detrended data.

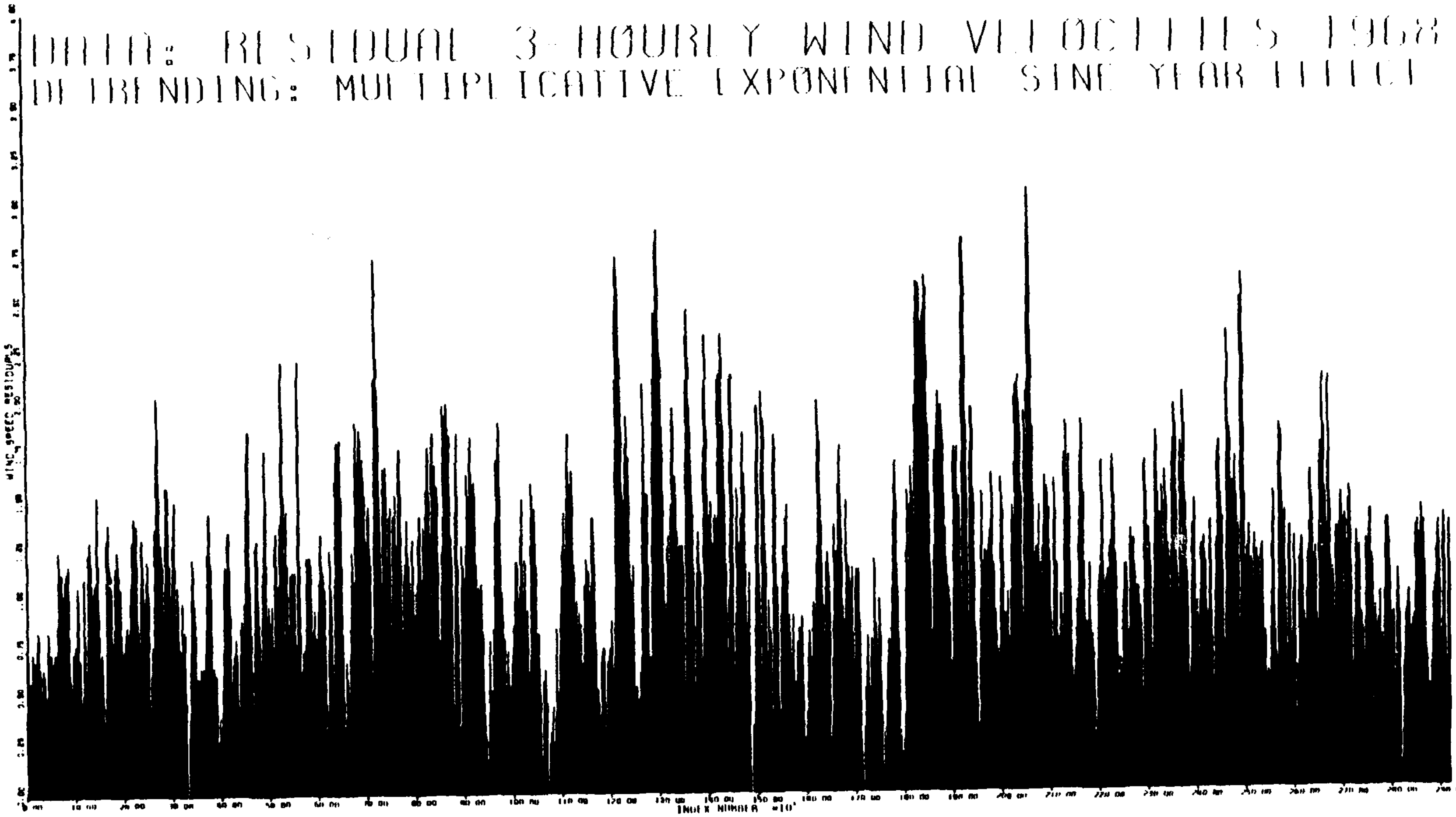


Figure IV.C.2a. 1968 Detrended data.

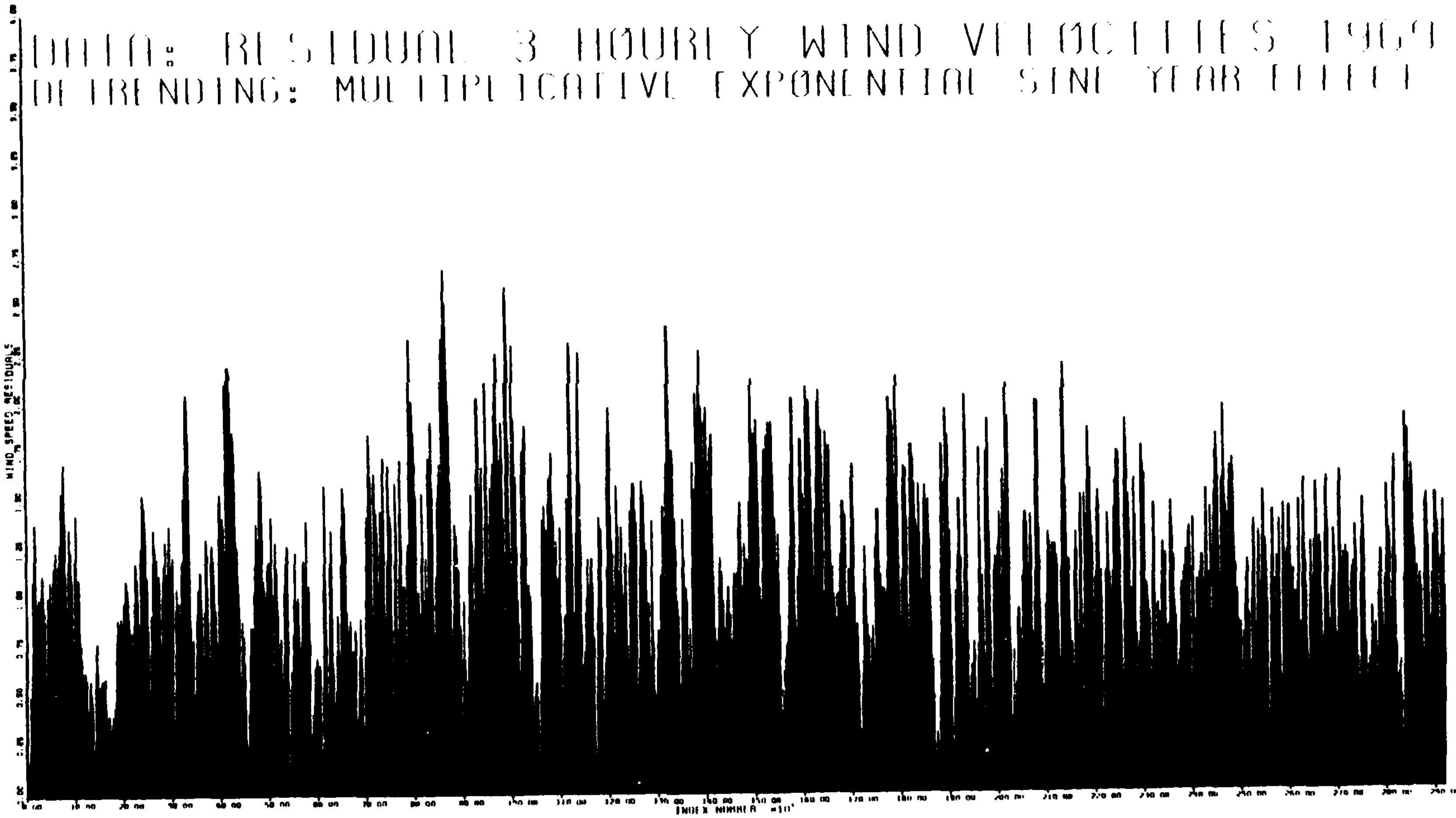


Figure IV.C.2o. 1969 Detrended data.

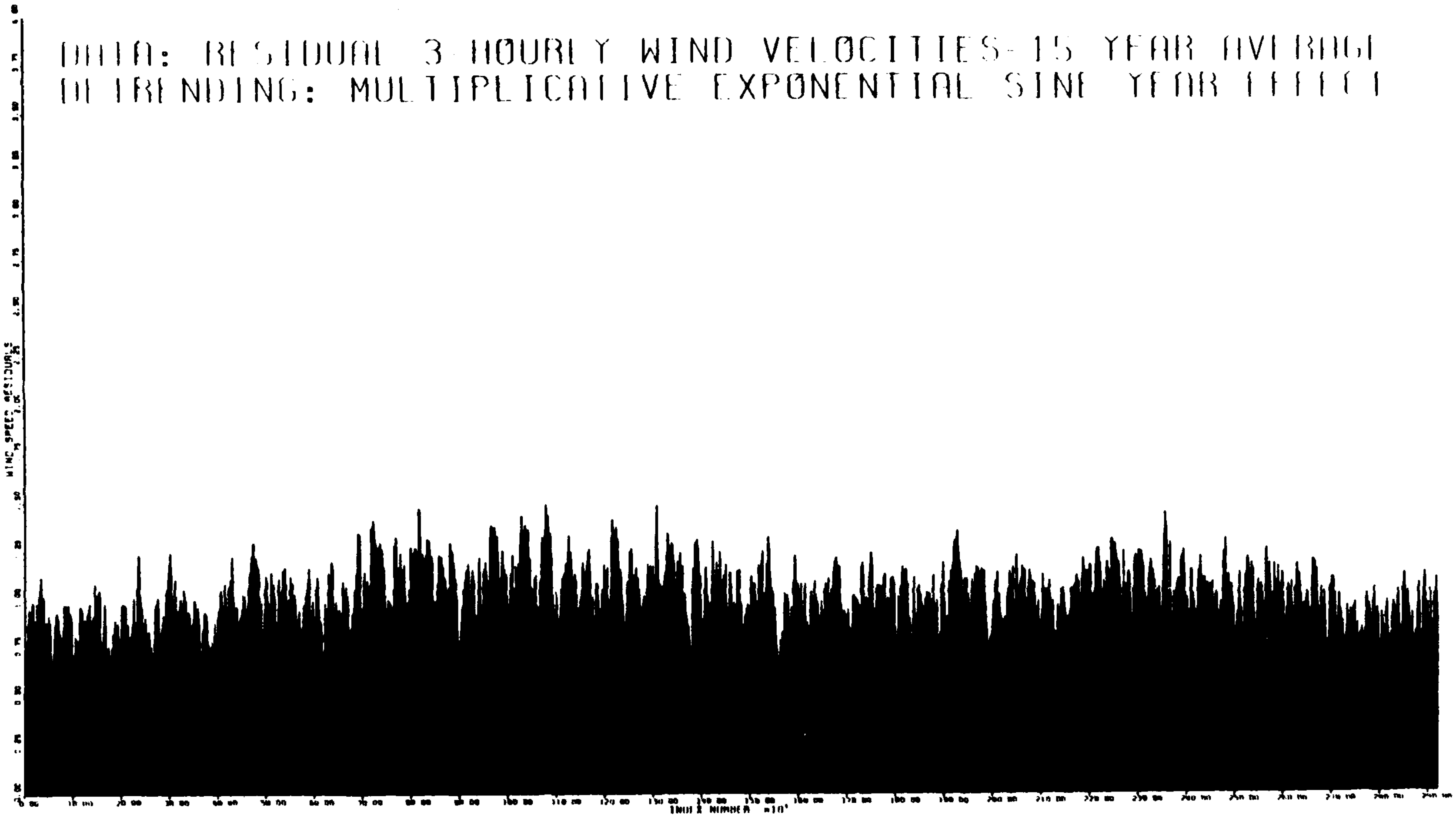


Figure IV.C.2p. Average path detrended (exponential sine)

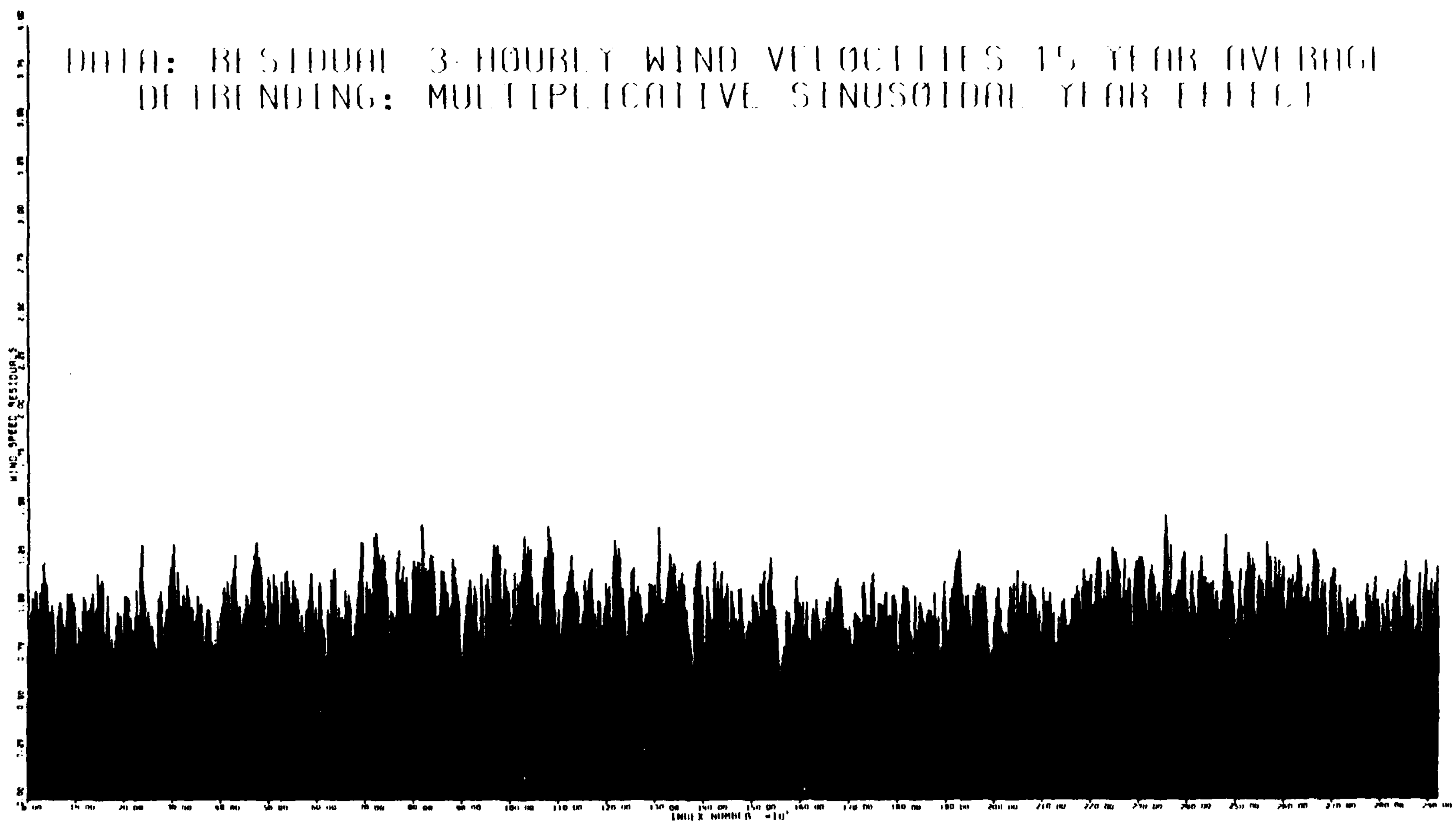


Figure IV.C.3. Average path detrended (sine).

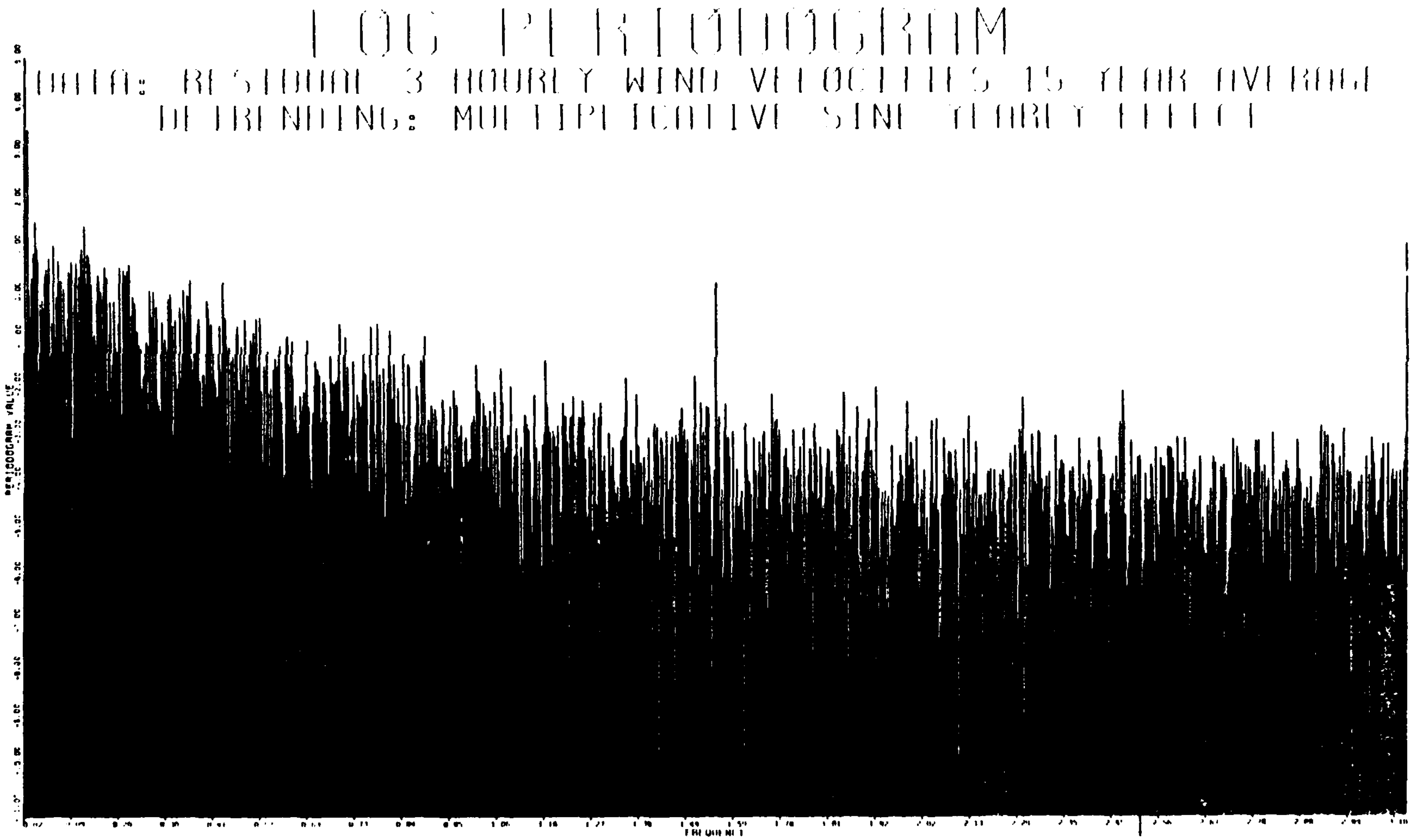


Figure IV.C.4. Log periodogram shows dominance of six-month cycle and presence of six and twelve hour cycles after removal of yearly cycle. Background spectrum resembles first-order autoregressive process.

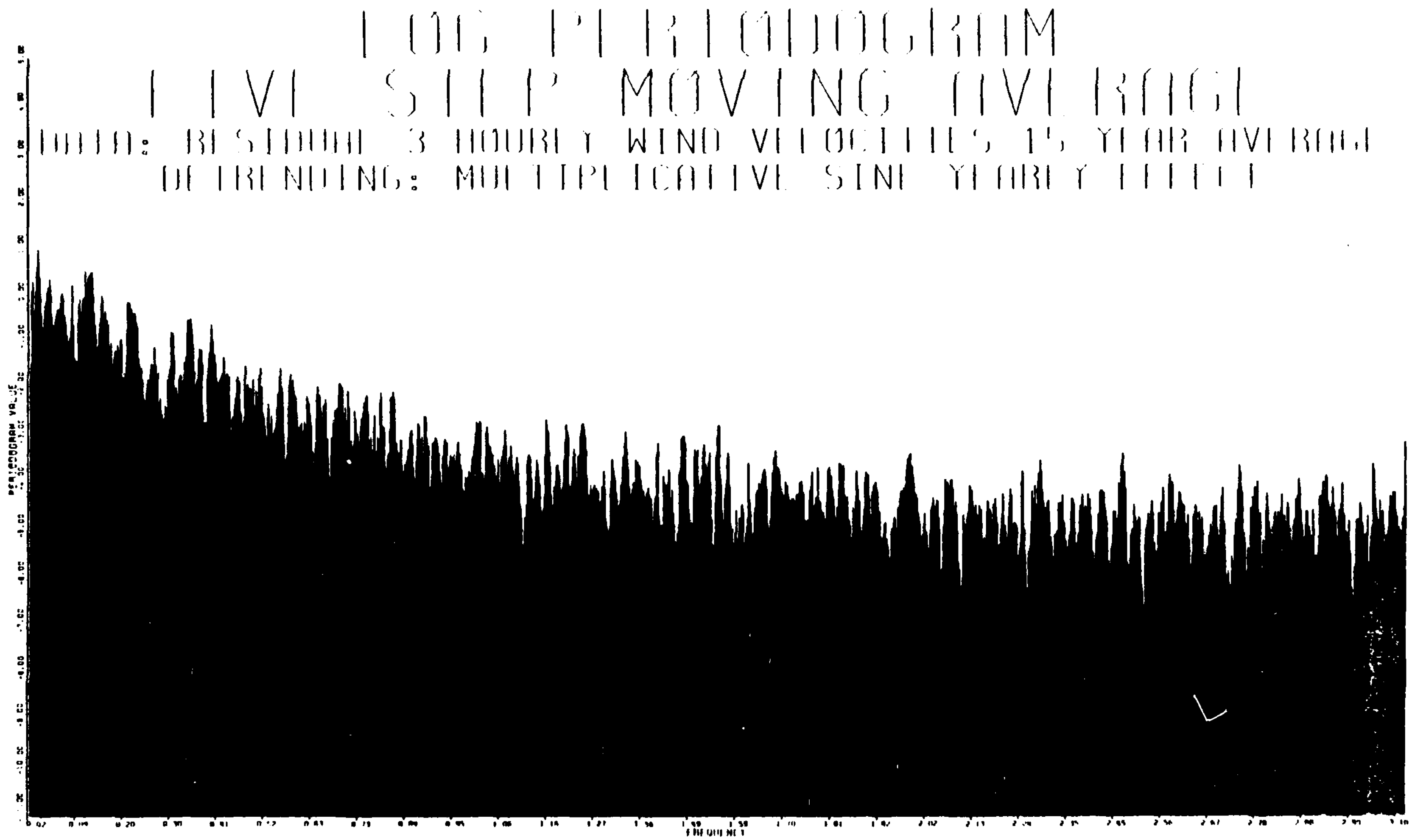

LOG PERIODOGRAM
FIVE STEP MOVING AVERAGE
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 15 YEAR AVERAGE
DETRENDING: MULTIPLICATIVE SINE YEARLY EFFECT
PERIODOGRAM VALUE
FREQUENCY

Figure IV.C.5.

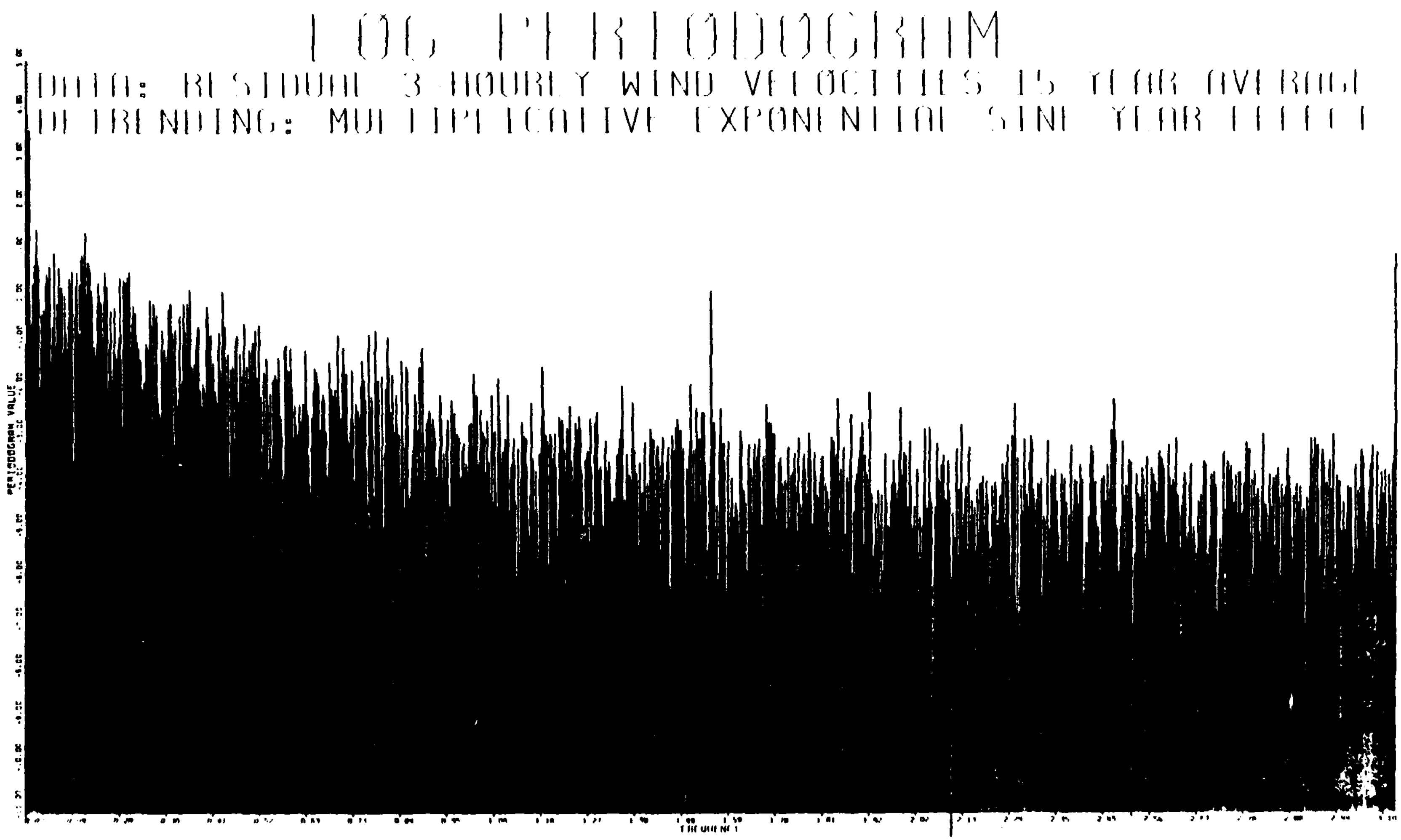


Figure IV.C.6. Log periodogram shows dominance of six-month cycle and presence of six and twelve hour cycles after removal of yearly cycle.

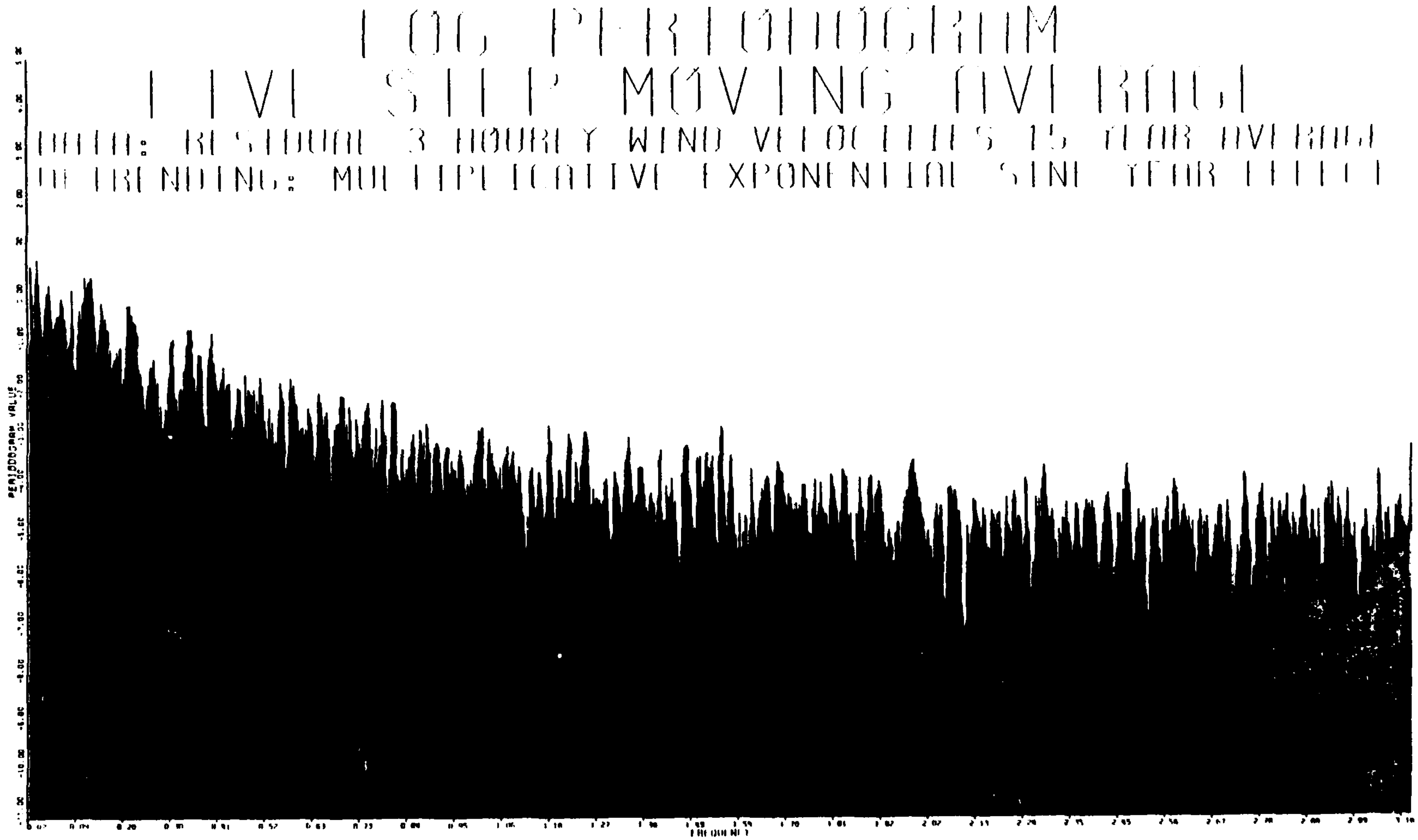

LOG PERIODOGRAM
FIVE STEP MOVING AVERAGE
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 15 YEAR AVERAGE
DETRENDING: MULTIPLICATIVE EXPONENTIAL SINE YEAR EFFECT
PERIODOGRAM VALUE
FREQUENCY

Figure IV.C.7.

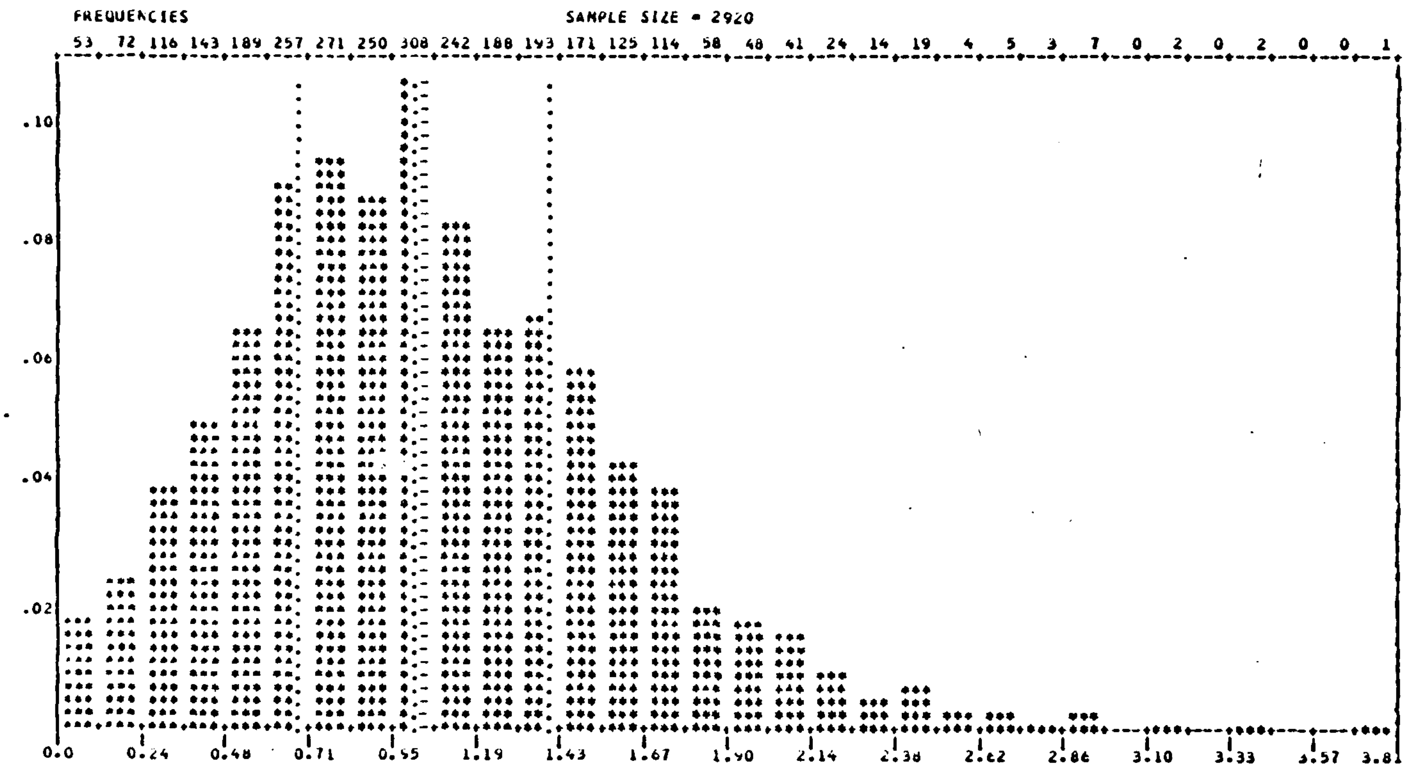
D. RESIDUAL PROCESS PROBABILITY STRUCTURE

Having removed the dominant seasonal effect from the data, it is possible to investigate the structure of the residual process ε_n in the model

$$X_n = \mu_n \varepsilon_n . \qquad \text{(IV.D.1)}$$

The two facets of the probability structure of the stationary process $\{\varepsilon_n\}$ which were addressed in Chapter II were the marginal distribution and the correlation structure. The residuals produced by dividing the raw data by the appropriate value of the mean were supplied to HISTF, a histogram and box plot routine developed at the Naval Postgraduate School. Histograms for each year and the entire data set were produced. These histograms are presented in Figures IV.D.1a through IV.D.1p. The shape of the histograms is consistent over the years and indicates that a Gamma distribution is appropriate for modeling the innovative factors. The parameter k can be estimated as the reciprocal of the coefficient of variation squared (see equation II.B.4.16). The estimated value of k for each year is given in Table IV.D.1.

A careful examination of the statistics associated with the histogram will reveal that the values for the skewness and kurtosis are low compared to the theoretical values for the Gamma distribution, namely $2/\sqrt{k}$ and $6/k$, respectively. However, this is not unexpected when one recalls the



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.04000	VARIANCE	.273113	M3	.851364E-01	MINIMUM	.0
MEDIAN	.950000	STD DEV	.522602	M4	.279497	.10 QUANTILE	.400000
TRIMEAN	1.01000	COEF VAR	0.5025	SKEWNESS	0.5965	.25 QUANTILE (HINGE)	.680000
MIDMEAN	.999651	MEAN DEV	.410178	KURTOSIS	0.7471	.50 QUANTILE (MEDIAN)	.950000
MIDRANGE	1.90500	RANGE	3.81000	BETA1	.850489E-01	.75 QUANTILE (HINGE)	1.38000
		MIDSPREAD	.700000	BETA2	.279268	.90 QUANTILE	1.70000
						MAXIMUM	3.81000

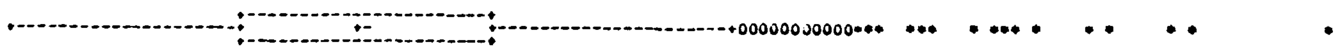
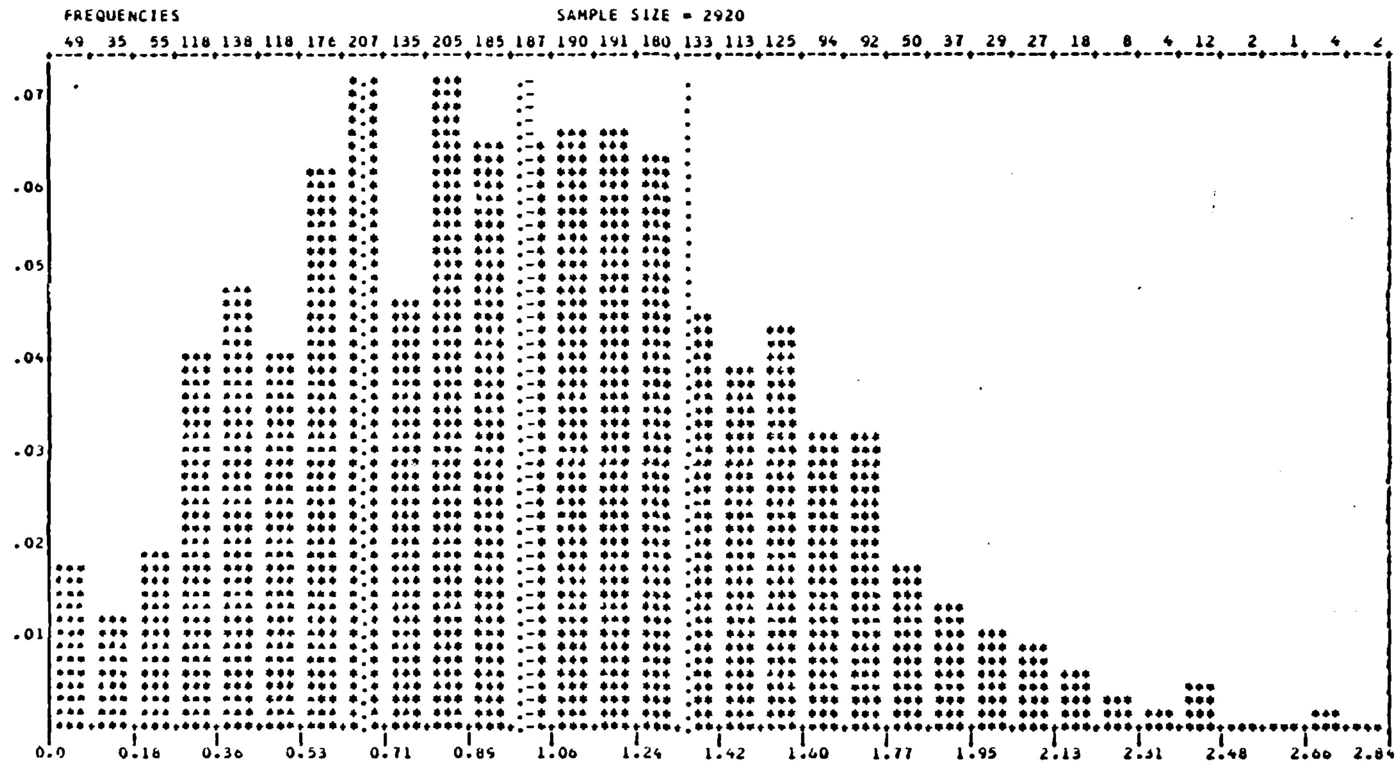


Figure IV.D.1a. Histogram and boxplot of 1955 3-hourly wind velocity residuals after detrending with one-year-harmonic multiplicative exponential sine.



CENTRAL TENDENCY	
MEAN	1.01776
MEDIAN	.550000
TRIMEAN	.997500
MIDMEAN	.993521
MIDRANGE	1.42000

SPREAD	
VARIANCE	.248681
STD DEV	.498679
COEF VAR	0.4900
MEAN DEV	.405271
RANGE	2.84000
MIDSPREAD	.710000

HIGHER CENTRAL MOMENTS	
M3	.410465E-01
M4	.175130
SKEWNESS	0.3310
KURTOSIS	-0.1681
BETA1	.410044E-01
BETA2	.175017

DISTRIBUTION	
MINIMUM	.0
.10 QUANTILE	.380000
.25 QUANTILE (HINGE)	.650000
.50 QUANTILE (MEDIAN)	.550000
.75 QUANTILE (HINGE)	1.36000
.90 QUANTILE	1.66000
MAXIMUM	2.84000

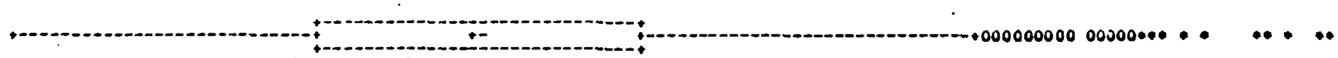
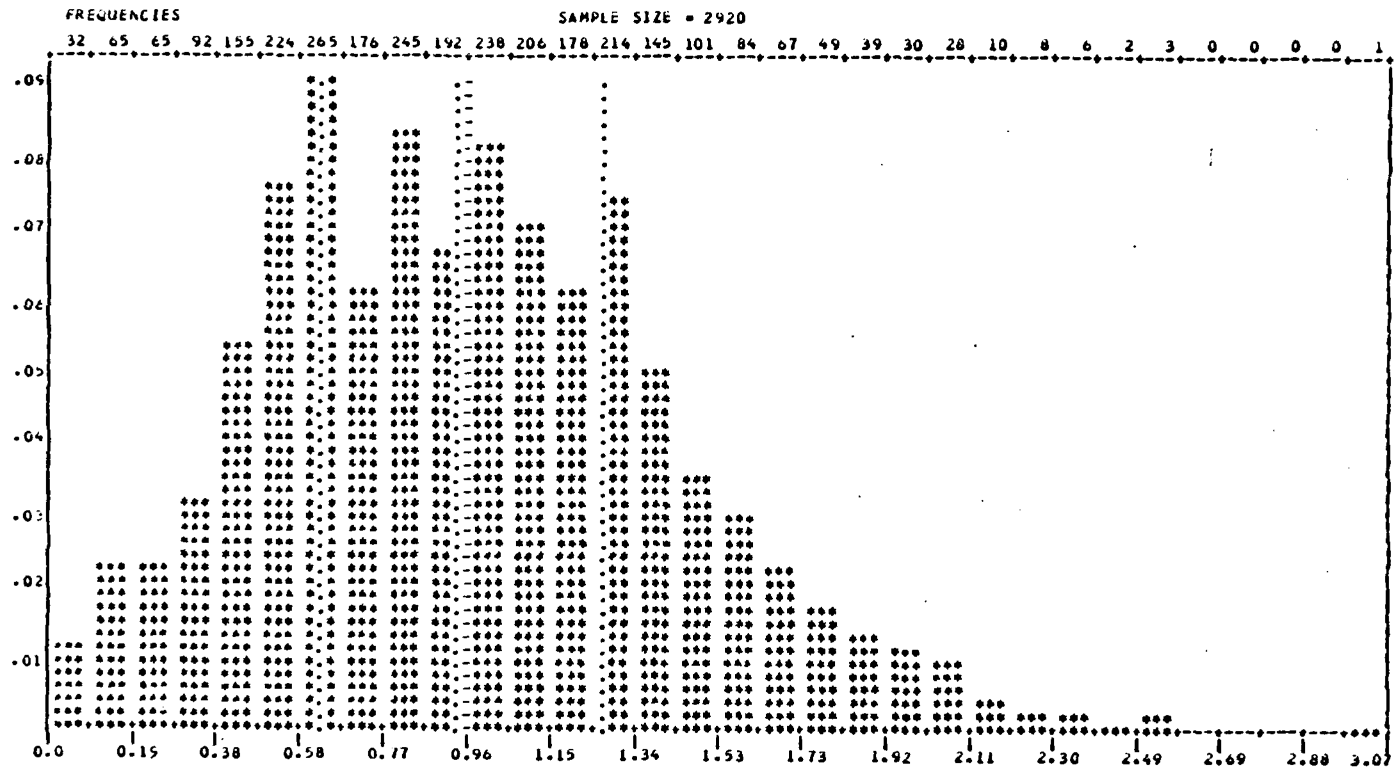


Figure IV.D.1b. Histogram and boxplot of 1956 detrended data.



CENTRAL TENDENCY	
MEAN	.961973
MEDIAN	.930000
TRIMEAN	.935000
MIDMEAN	.922521
MIDRANGE	1.53500

SPREAD	
VARIANCE	.211055
STD DEV	.459407
COEF VAR	0.4776
MEAN DEV	.371555
RANGE	3.07000
MIDSPREAD	.660001

HIGHER CENTRAL MOMENTS	
M3	.406649E-01
M4	.132506
SKEWNESS	0.4194
KURTOSIS	-0.0253
BETA1	.406231E-01
BETA2	.132416

DISTRIBUTION	
MINIMUM	.0
.10 QUANTILE	.410000
.25 QUANTILE (HINGE)	.610000
.50 QUANTILE (MEDIAN)	.930000
.75 QUANTILE (HINGE)	1.27000
.90 QUANTILE	1.57000
MAXIMUM	3.07000

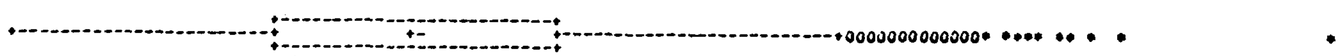
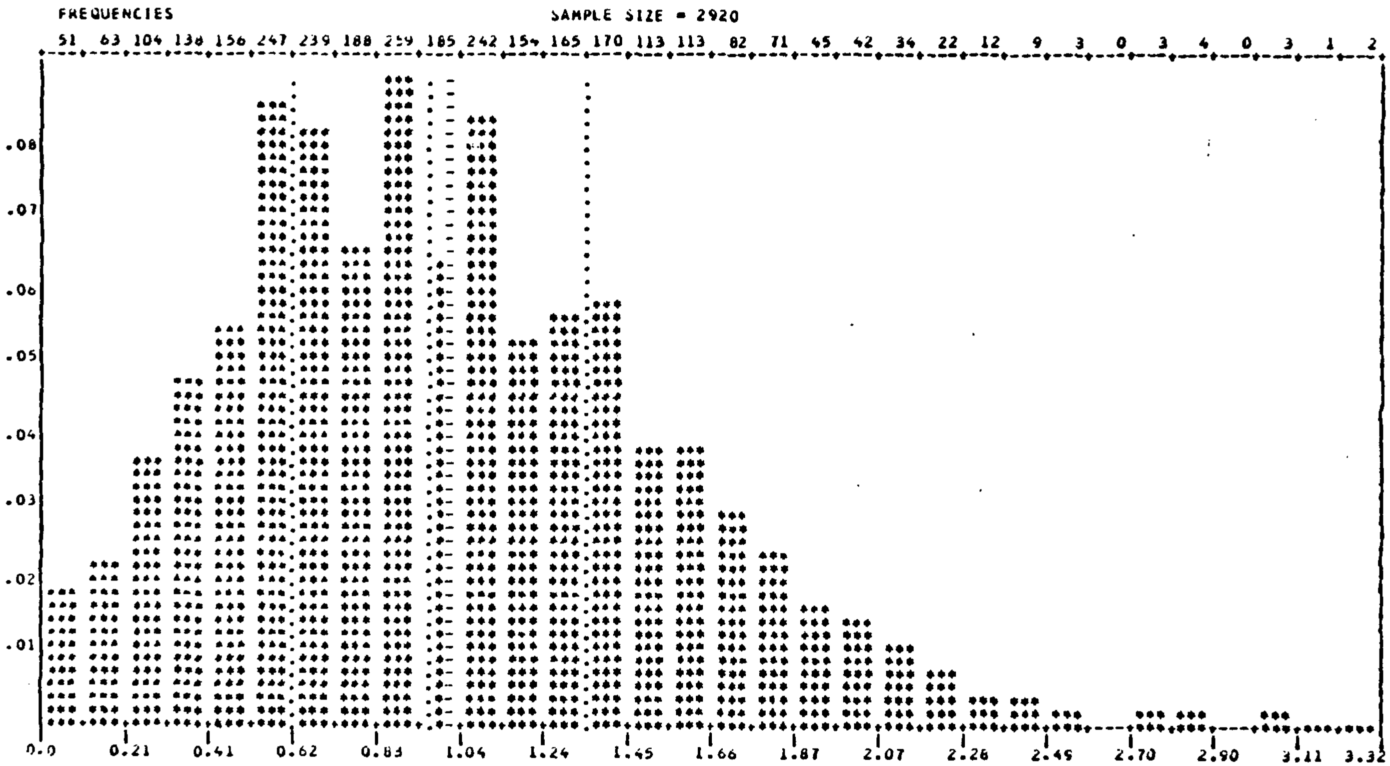


Figure IV.D.1c. Histogram and boxplot of 1957 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.00267	VARIANCE	.273341	M3	.815271E-01	MINIMUM	.0
MEDIAN	.540000	STD DEV	.522820	M4	.245332	.10 QUANTILE	.370000
TRIMEAN	.957500	COEF VAR	0.5214	SKEWNESS	0.5705	.25 QUANTILE (HINGE)	.610000
MIDMEAN	.955562	MEAN DEV	.419164	KURTOSIS	0.2836	.50 QUANTILE (MEDIAN)	.540000
MIDRANGE	1.66000	RANGE	3.32000	BETA1	.814434E-01	.75 QUANTILE (HINGE)	1.34000
		MIDSPREAD	.730000	BETA2	.245149	.90 QUANTILE	1.70000
						MAXIMUM	3.32000

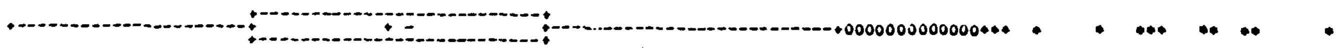
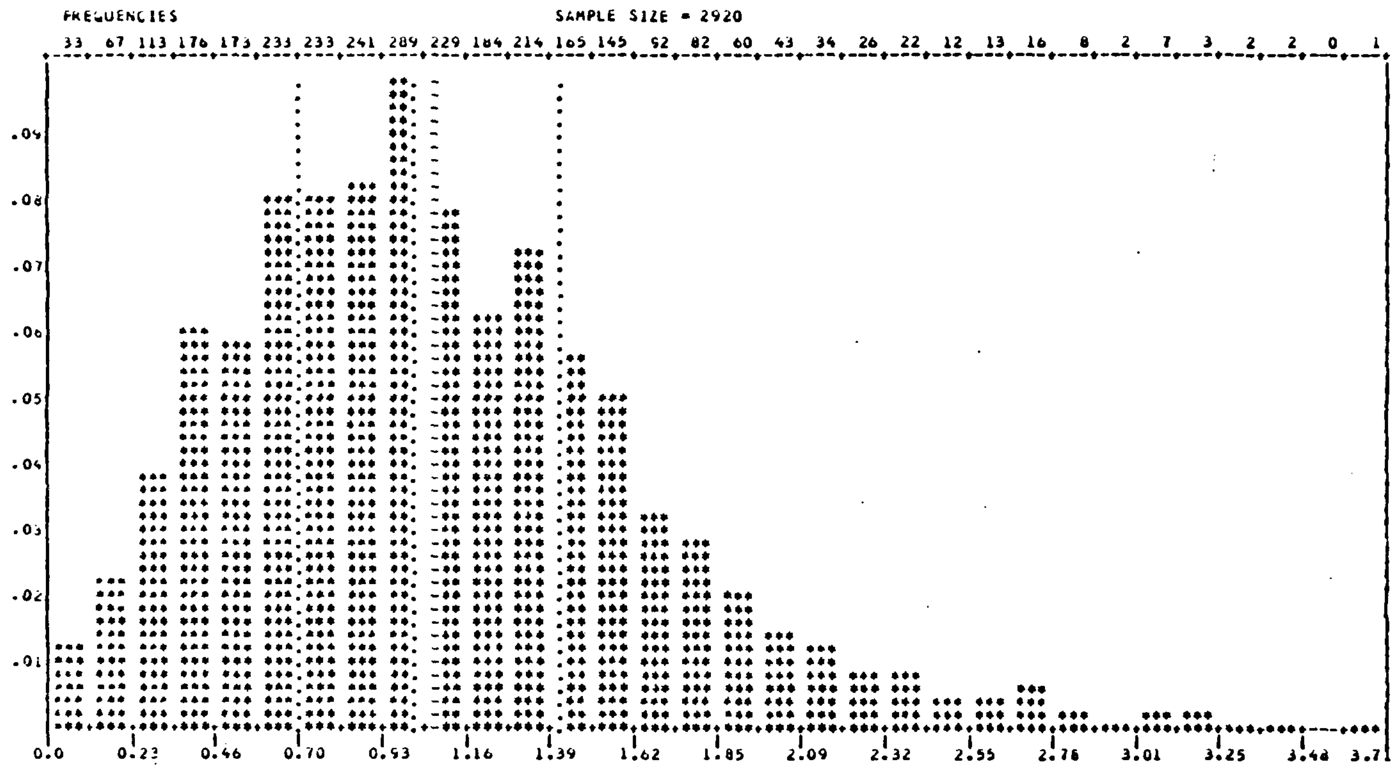


Figure IV.D.1d. Histogram and boxplot of 1958 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.07065	VARIANCE	.313908	M3	.139281	MINIMUM	.0
MEDIAN	1.00000	STD DEV	.560275	M4	.389296	.10 QUANTILE	.400000
TRIMEAN	1.01750	COEF VAR	0.5233	SKEWNESS	0.7919	.25 QUANTILE (HINGE)	.670000
MIDMEAN	1.01560	MEAN DEV	.435055	KURTOSIS	0.9507	.50 QUANTILE (MEDIAN)	1.00000
MIDRANGE	1.85500	RANGE	3.71000	BETA1	.139138	.75 QUANTILE (HINGE)	1.40000
		MIDSPREAD	.730000	BETA2	.388965	.90 QUANTILE	1.80000
						MAXIMUM	3.71000

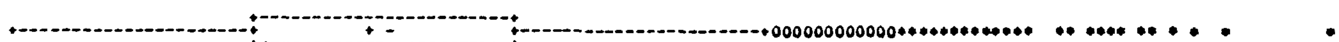


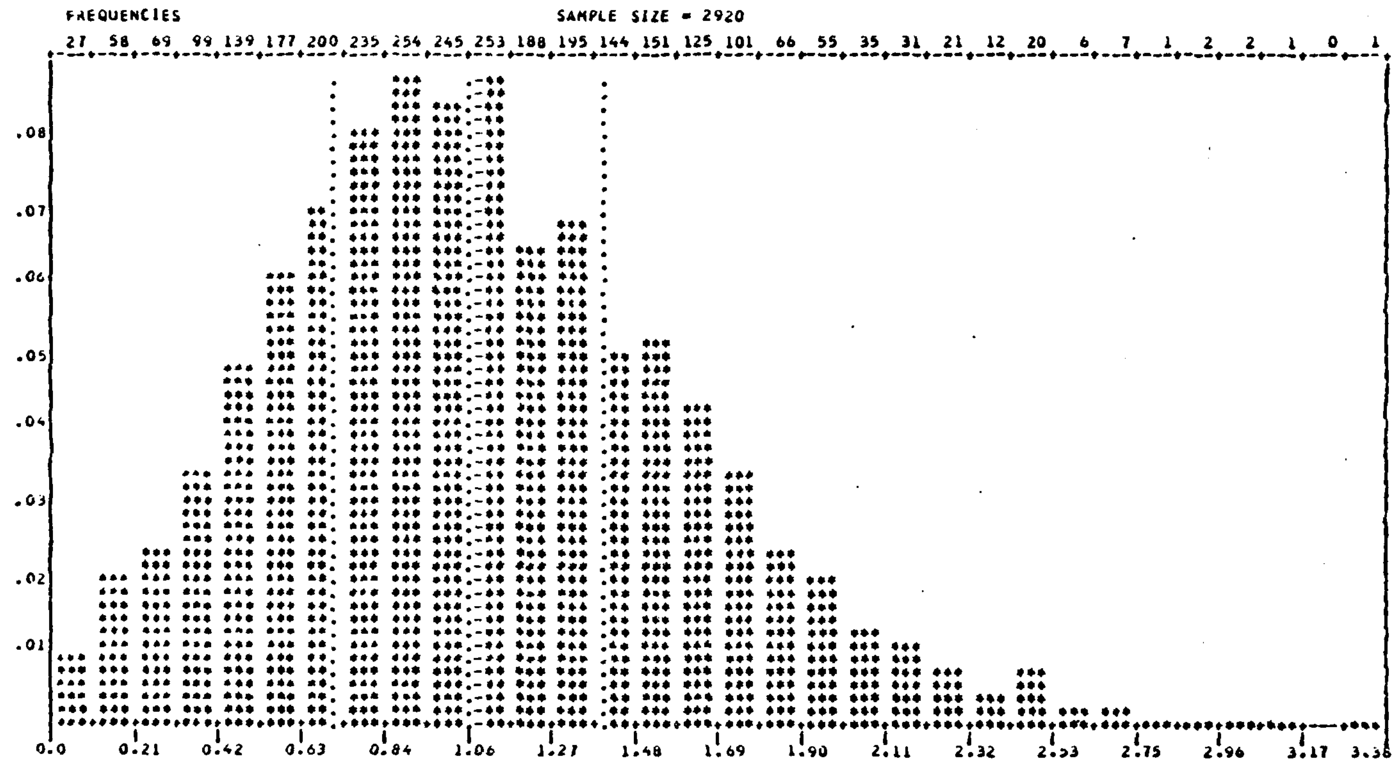
Figure IV.D.1e. Histogram and boxplot of 1959 detrended data.

AD-A115 754 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 12/1
EXTENSION OF SOME MODELS FOR POSITIVE-VALUED TIME SERIES.(U)
MAR 82 D K HUGUS

UNCLASSIFIED NL

4 OF 4
AD A
115754

END
DATE
FILMED
7 82
DTIC



CENTRAL TENDENCY	
MEAN	1.08395
MEDIAN	1.04000
TRIMEAN	1.04750
MIDMEAN	1.04376
MIDRANGE	1.69000

SPREAD	
VARIANCE	.264269
STD DEV	.514071
COEF VAR	0.4743
MEAN DEV	.406818
RANGE	3.38000
MIDSPREAD	.690000

HIGHER CENTRAL MOMENTS	
M3	.694286E-01
M4	.227356
SKEWNESS	0.5111
KURTOSIS	0.2555
BETA1	.693573E-01
BETA2	.227188

DISTRIBUTION	
MINIMUM	.0
.10 QUANTILE	.460000
.25 QUANTILE (HINGE)	.710000
.50 QUANTILE (MEDIAN)	1.04000
.75 QUANTILE (HINGE)	1.40000
.90 QUANTILE	1.77000
MAXIMUM	3.38000

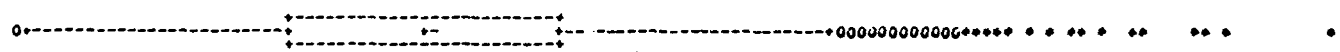
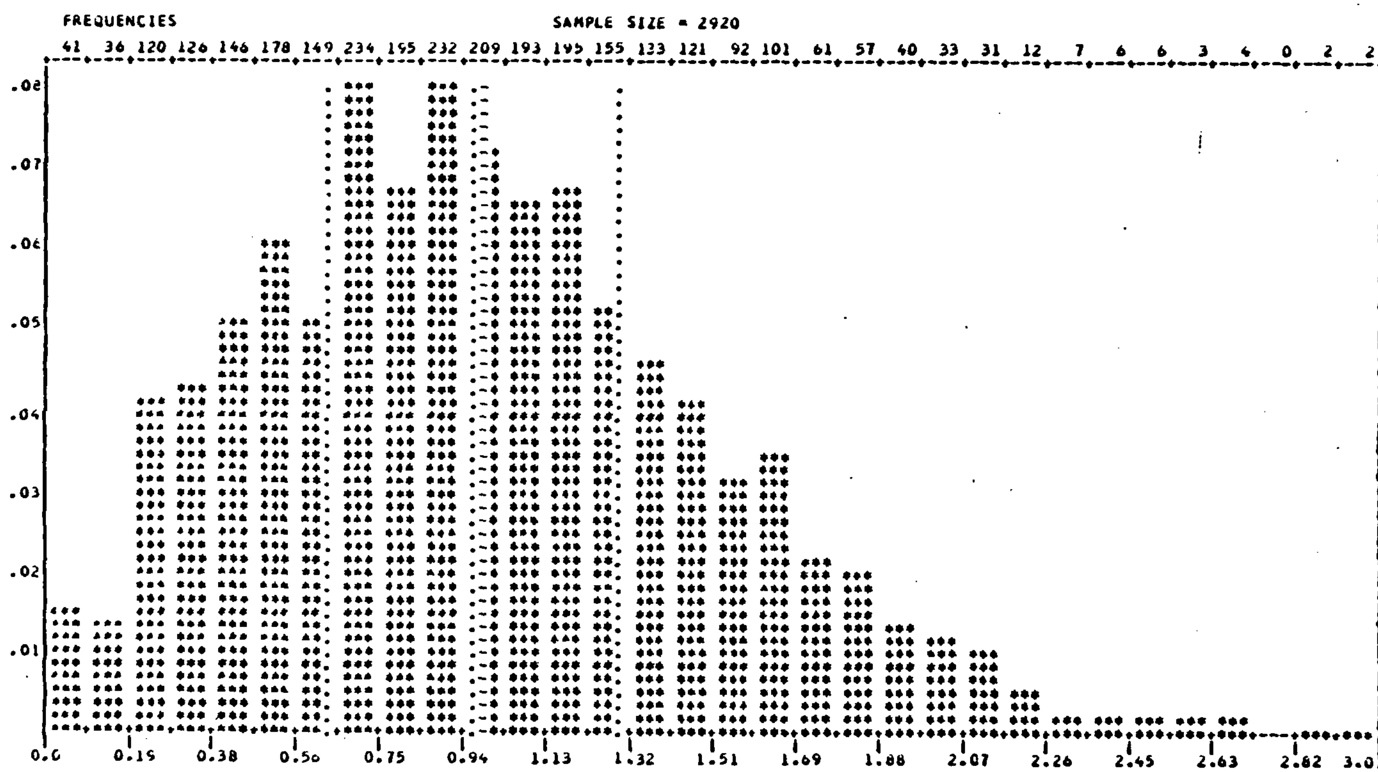


Figure IV.D.1f. Histogram and boxplot of 1960 detrended data.



CENTRAL TENDENCY	
MEAN	.586634
MEDIAN	.550000
TRIMEAN	.555000
MIDMEAN	.550431
MIDRANGE	1.50500

SPREAD	
VARIANCE	.251582
STD DEV	.501579
COEF VAR	0.5084
MEAN DEV	.400962
RANGE	3.01000
MIDSPREAD	.680000

HIGHER CENTRAL MOMENTS	
M3	.606103E-01
M4	.193191
SKEWNESS	0.4803
KURTOSIS	0.0523
BETA1	.605480E-01
BETA2	.193056

DISTRIBUTION	
MINIMUM	.0
.10 QUANTILE	.350000
.25 QUANTILE (HINGE)	.620000
.50 QUANTILE (MEDIAN)	.550000
.75 QUANTILE (HINGE)	1.30000
.90 QUANTILE	1.67000
MAXIMUM	3.01000

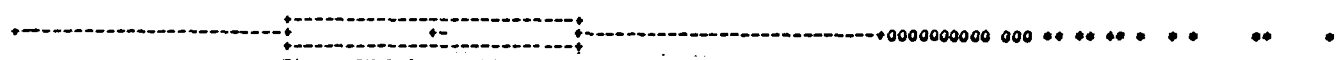
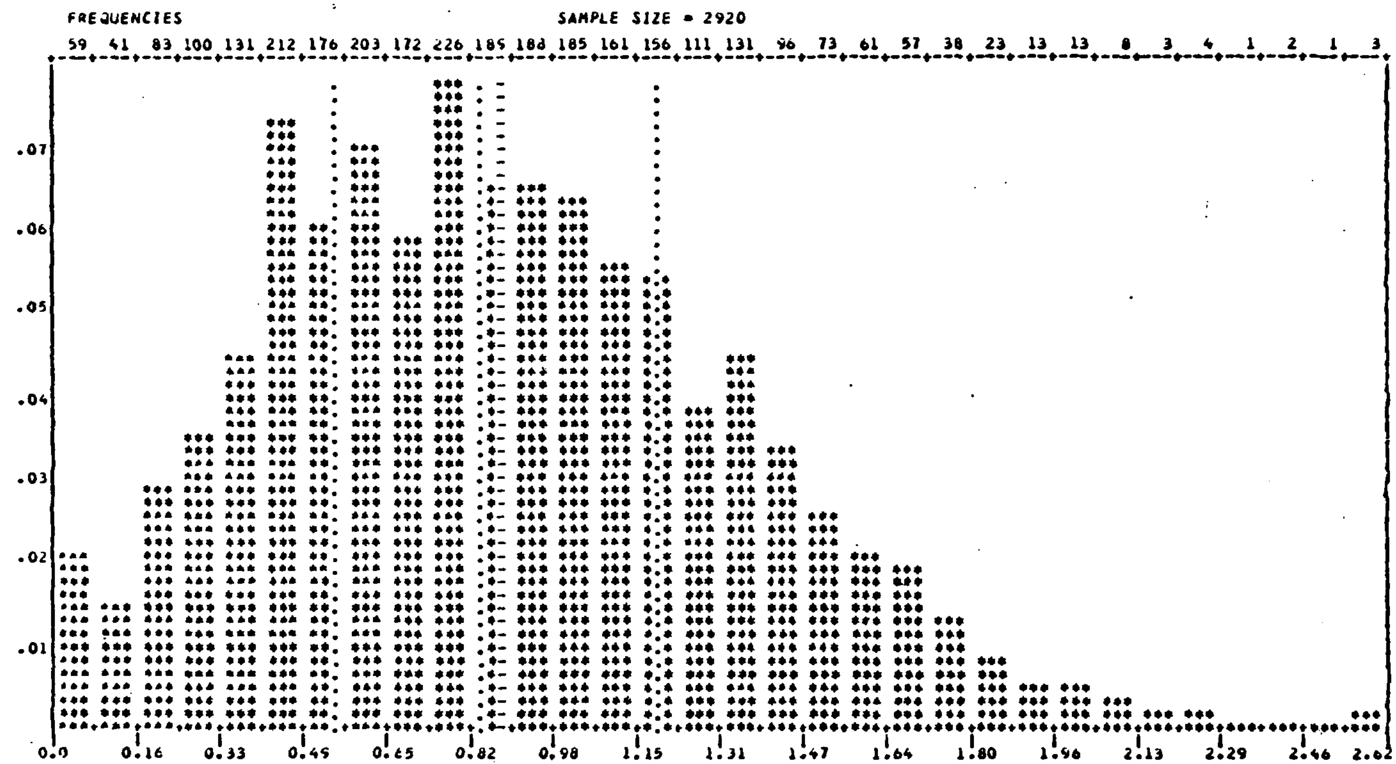


Figure IV.D.1g. Histogram and boxplot of 1961 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	.879514	VARIANCE	.199967	M3	.3523396E-01	MINIMUM	.0
MEDIAN	.840000	STD DEV	.447177	M4	.115358	.10 QUANTILE	.330000
TRIMEAN	.852500	COEF VAR	0.5084	SKEWNESS	0.3940	.25 QUANTILE (HINGE)	.550000
MIDMEAN	.850096	MEAN DEV	.362931	KURTOSIS	-0.1151	.50 QUANTILE (MEDIAN)	.640000
MIDRANGE	1.31000	RANGE	2.62000	BETA1	.351977E-01	.75 QUANTILE (HINGE)	1.18000
		MIDSPREAD	.630001	BETA2	.115282	.90 QUANTILE	1.48000
						MAXIMUM	2.62000

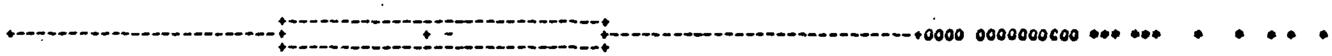
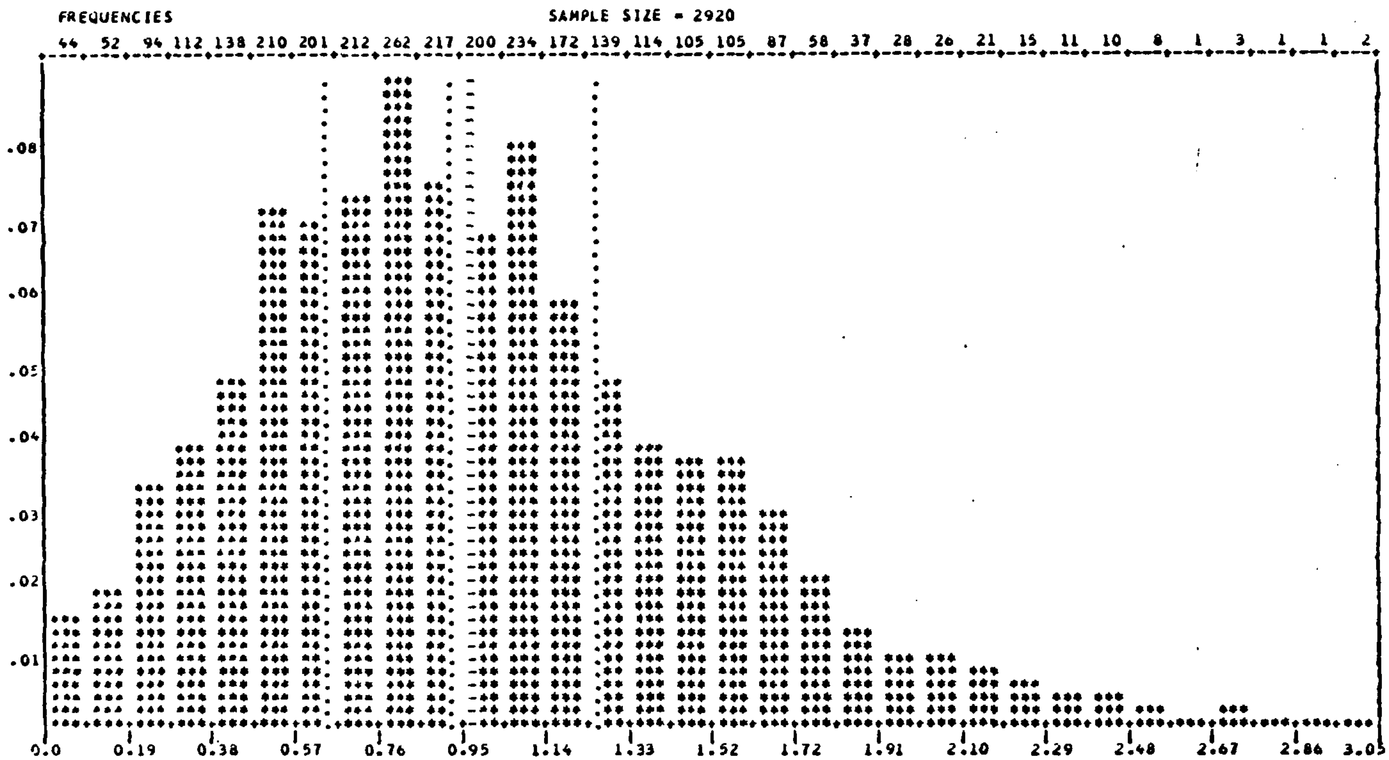


Figure IV.D.1h. Histogram and boxplot of 1962 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	.971726	VARIANCE	.244328	M3	.723460E-01	MINIMUM	.0
MEDIAN	.920000	STD DEV	.494296	M4	.200018	.10 QUANTILE	.380000
TRIMEAN	.932500	COEF VAR	0.5087	SKEWNESS	0.5990	.25 QUANTILE (HINGE)	.620000
MIDMEAN	.925397	MEAN DEV	.389541	KURTOSIS	0.3506	.50 QUANTILE (MEDIAN)	.920000
MIDRANGE	1.52500	RANGE	3.05000	BETA1	.722718E-01	.75 QUANTILE (HINGE)	1.27000
		MIDSPREAD	.650001	BETA2	.199867	.90 QUANTILE	1.65000
						MAXIMUM	3.05000

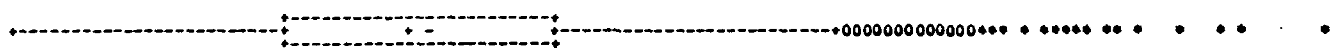
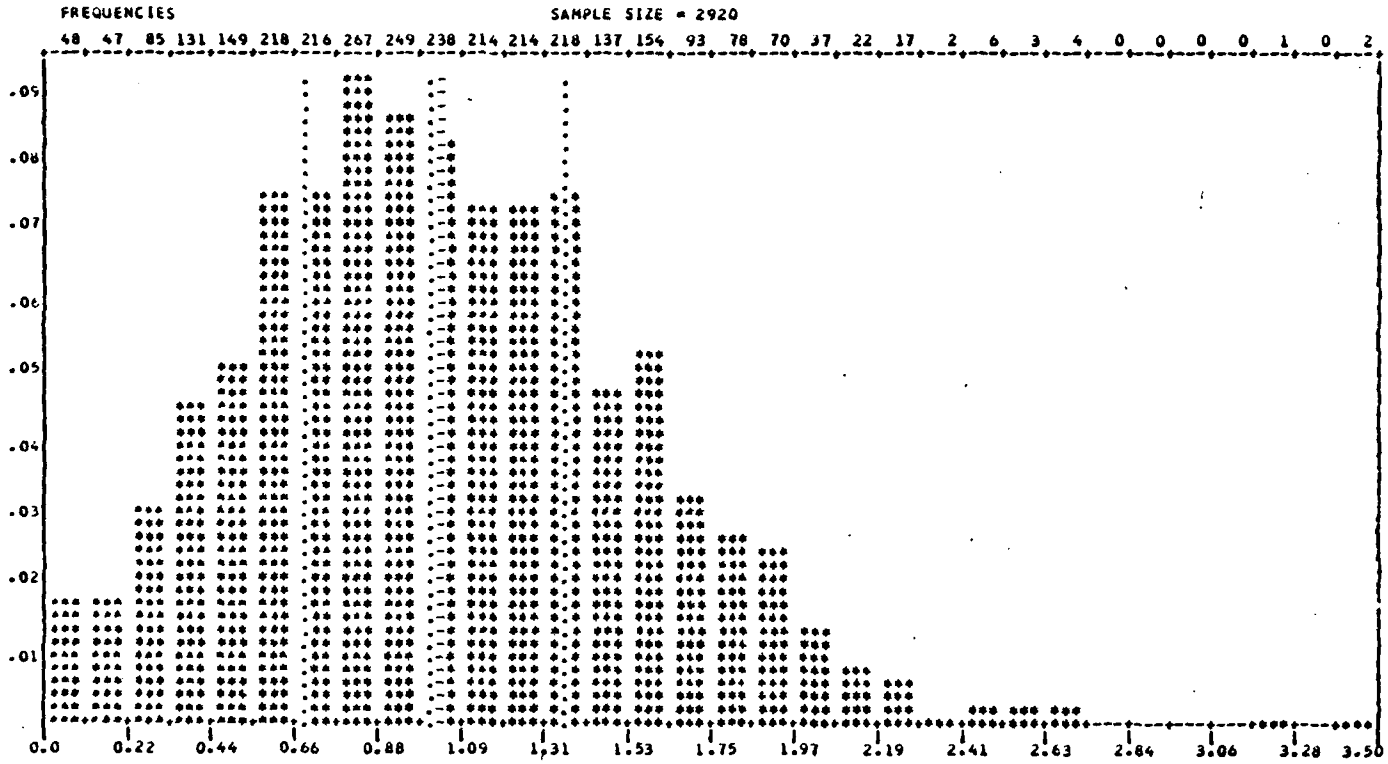


Figure IV.D.11. Histogram and boxplot of 1963 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.04109	VARIANCE	.241341	M3	.418640E-01	MINIMUM	.0
MEDIAN	1.00500	STD DEV	.491264	M4	.183669	.10 QUANTILE	.420000
TRIMEAN	1.01500	COEF VAR	0.4719	SKEWNESS	0.3531	.25 QUANTILE (HINGE)	.680000
MIDMEAN	1.01812	MEAN DEV	.396349	KURTOSIS	0.1534	.50 QUANTILE (MEDIAN)	1.00500
MIDRANGE	1.75000	RANGE	3.50000	BETA1	.418210E-01	.75 QUANTILE (HINGE)	1.37000
		MIDSPREAD	.690000	BETA2	.183537	.90 QUANTILE	1.69000
						MAXIMUM	3.50000

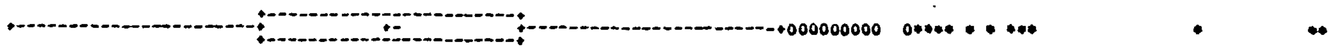
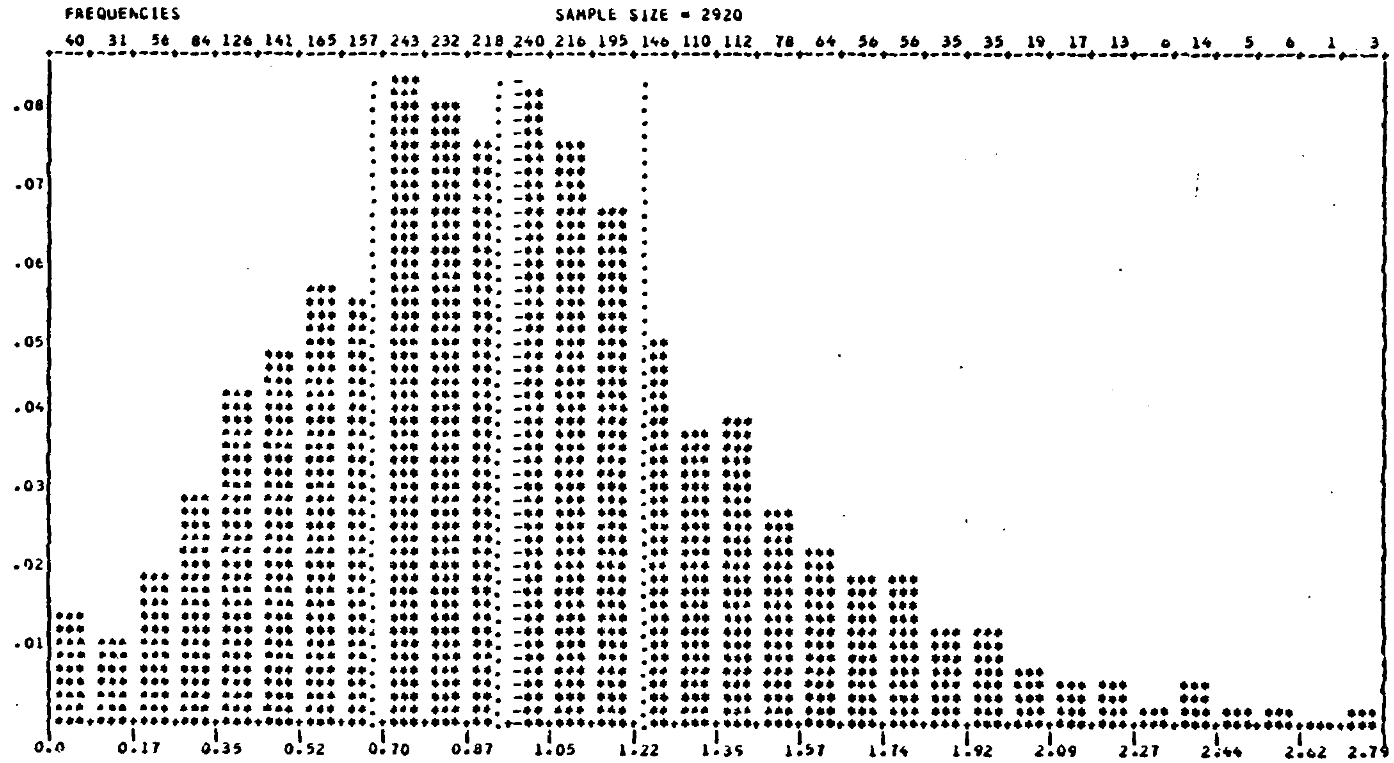


Figure IV.D.1j. Histogram and boxplot of 1964 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	.583517	VARIANCE	.223665	M3	.590470E-01	MINIMUM	.0
MEDIAN	.940000	STD DEV	.472932	M4	.170820	.10 QUANTILE	.410000
TRIMEAN	.947500	COEF VAR	0.4609	SKEWNESS	0.5582	.25 QUANTILE (HINGE)	.660000
MIDMEAN	.947658	MEAN DEV	.368243	KURTOSIS	0.4146	.50 QUANTILE (MEDIAN)	.540000
MIDRANGE	1.39500	RANGE	2.79000	BETA1	.589864E-01	.75 QUANTILE (HINGE)	1.25000
		MIDSPREAD	.590000	BETA2	.170668	.90 QUANTILE	1.62000
						MAXIMUM	2.79000

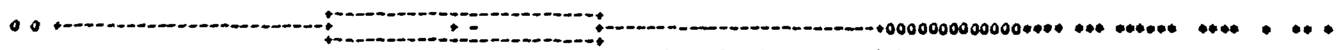
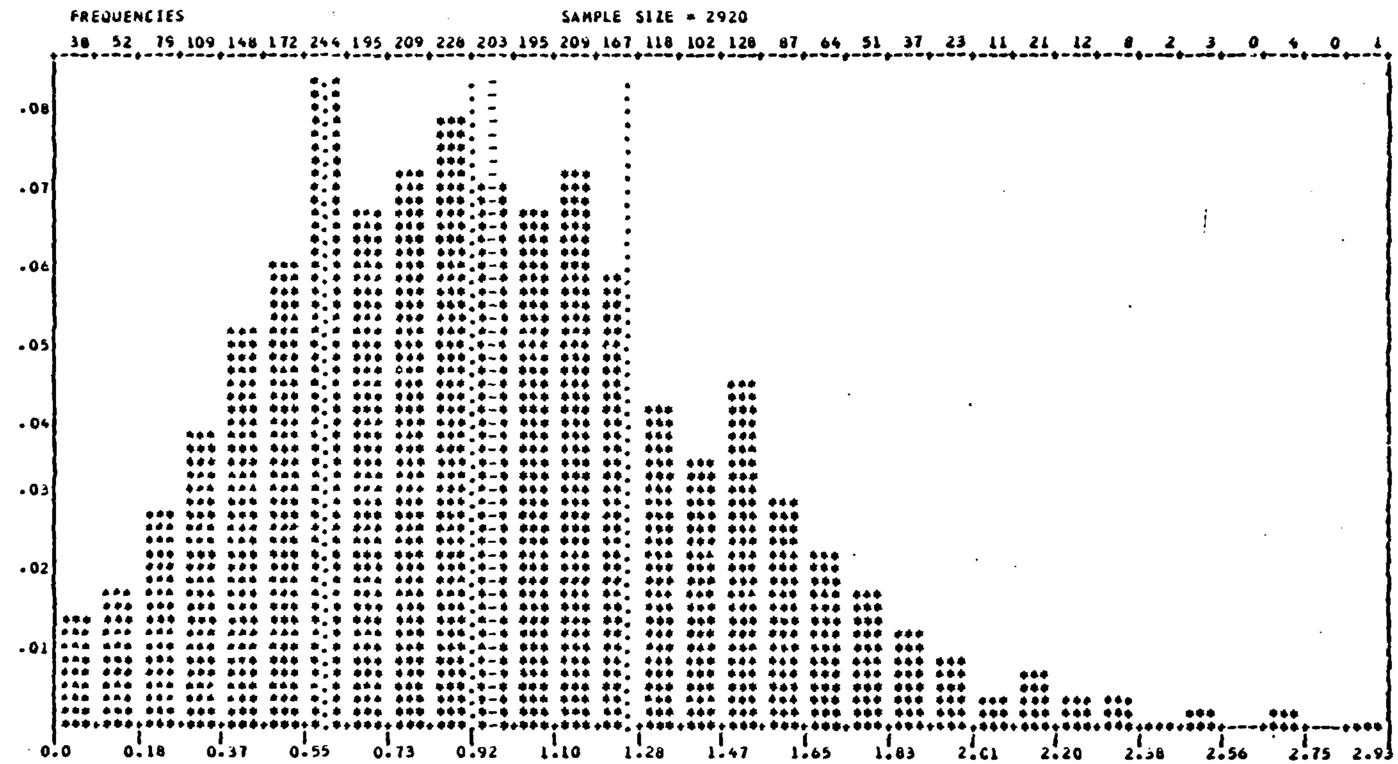


Figure IV.D.1k. Histogram and boxplot of 1965 detrended data.



CENTRAL TENDENCY	
MEAN	.552647
MEDIAN	.510000
TRIMEAN	.515000
MIDMEAN	.515548
MIDRANGE	1.46500

SPREAD	
VARIANCE	.224712
STD DEV	.474038
COEF VAR	0.4976
MEAN DEV	.380565
RANGE	2.93000
MIDSPREAD	.660000

HIGHER CENTRAL MOMENTS	
M3	.505893E-01
M4	.153513
SKEWNESS	0.4749
KURTOSIS	0.0401
BETA1	.505373E-01
BETA2	.153407

DISTRIBUTION	
MINIMUM	.0
.10 QUANTILE	.370000
.25 QUANTILE (HINGE)	.590000
.50 QUANTILE (MEDIAN)	.510000
.75 QUANTILE (HINGE)	1.25000
.90 QUANTILE	1.59000
MAXIMUM	2.93000

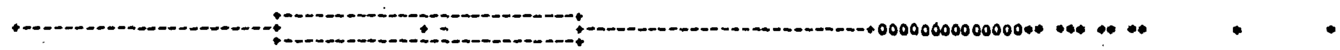
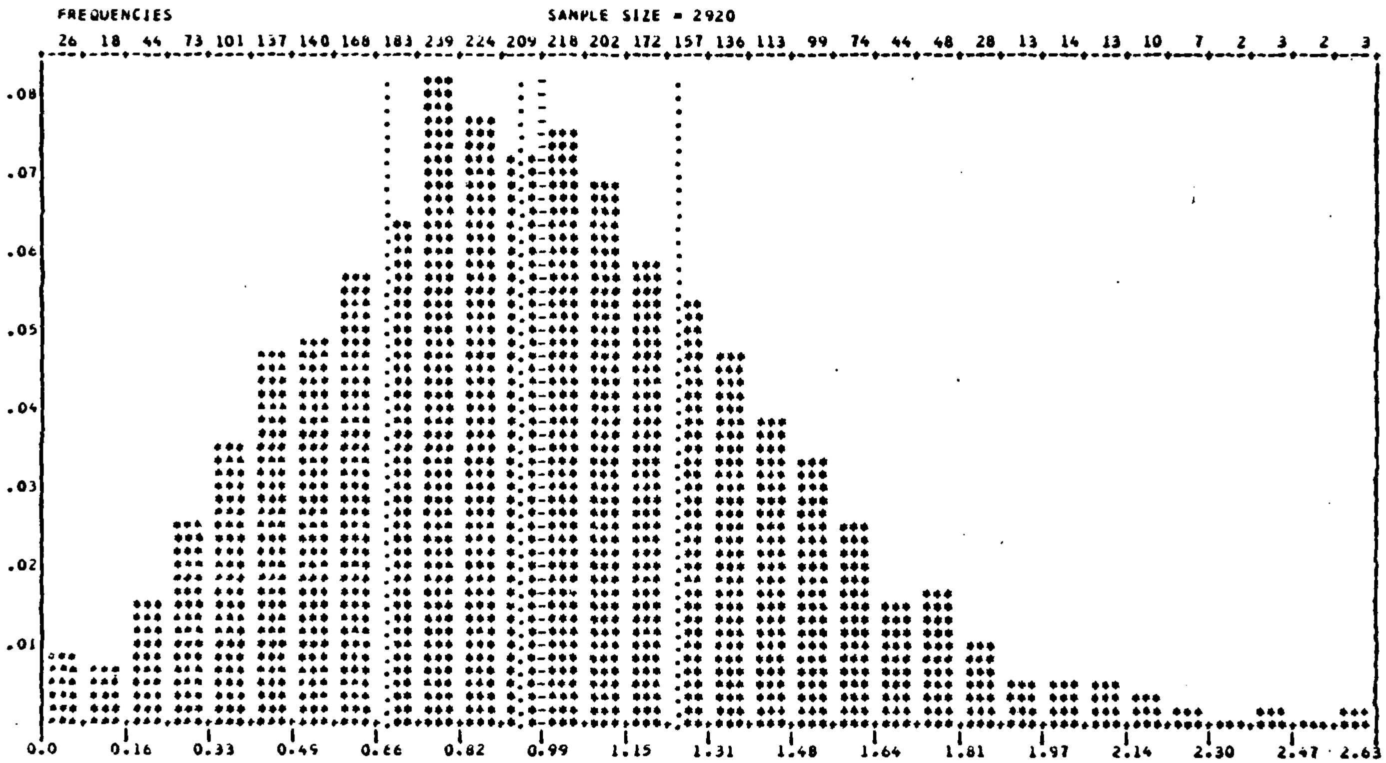


Figure IV.D.11. Histogram and boxplot of 1966 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	.971123	VARIANCE	.185584	M3	.307422E-01	MINIMUM	.0
MEDIAN	.940000	STD DEV	.430795	M4	.107399	.10 QUANTILE	.430000
TRIMEAN	.952500	COEF VAR	0.4436	SKEWNESS	0.3845	.25 QUANTILE (HINGE)	.670000
MIDMEAN	.948911	MEAN DEV	.343836	KURTOSIS	0.1183	.50 QUANTILE (MEDIAN)	.940000
MIDRANGE	1.31500	RANGE	2.63000	BETA1	.307106E-01	.75 QUANTILE (HINGE)	1.26000
		MIDSPREAD	.590001	BETA2	.107323	.90 QUANTILE	1.54000
						MAXIMUM	2.63000

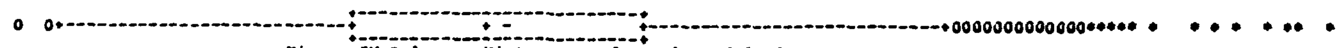
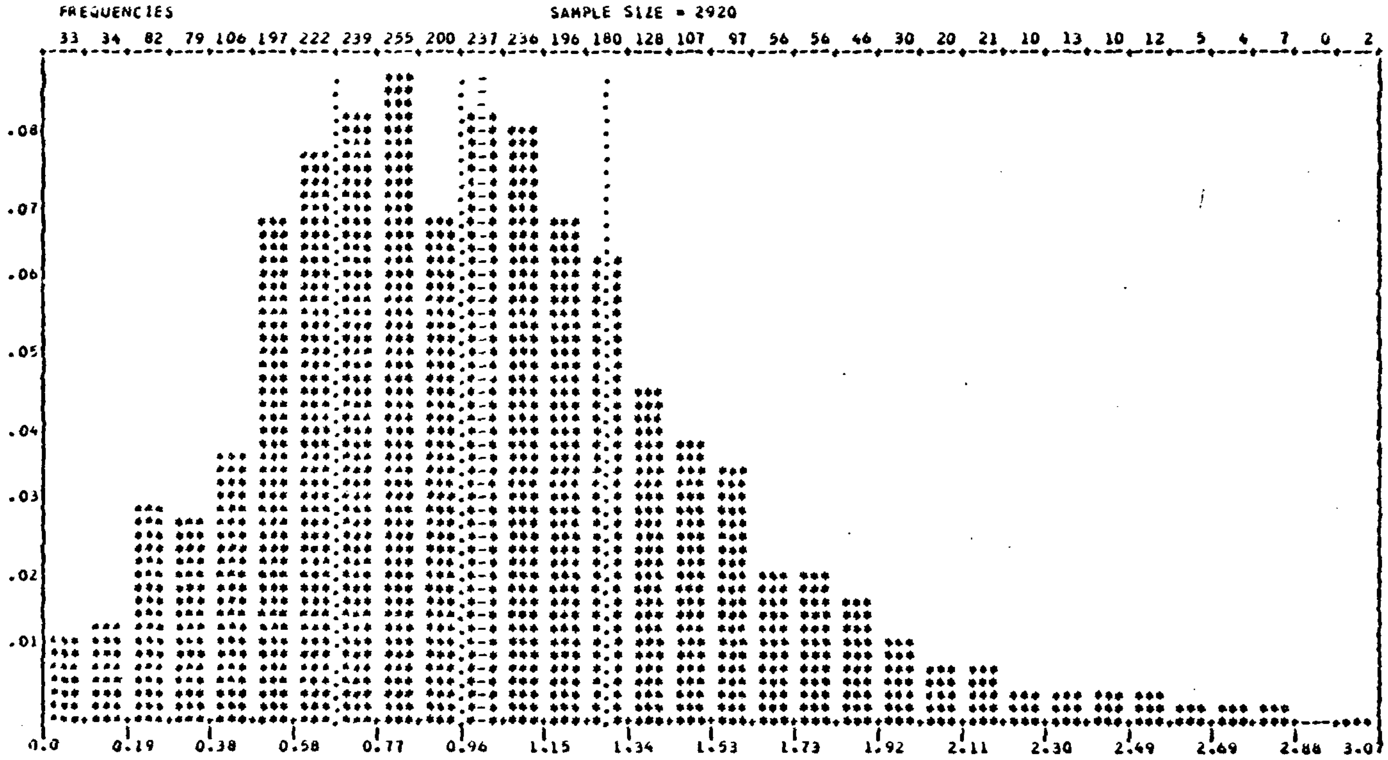


Figure IV.D.1m. Histogram and boxplot of 1967 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.00868	VARIANCE	.240131	M3	.829471E-01	MINIMUM	.0
MEDIAN	.960000	STD DEV	.490031	M4	.220236	.10 QUANTILE	.420000
TRIMEAN	.967499	COEF VAR	0.4858	SKEWNESS	0.7049	.25 QUANTILE (HINGE)	.670000
MIDMEAN	.963651	MEAN DEV	.381595	KURTOSIS	0.8194	.50 QUANTILE (MEDIAN)	.960000
MIDRANGE	1.53500	RANGE	3.07000	BETA1	.828619E-01	.75 QUANTILE (HINGE)	1.28000
		MIDSPREAD	.610000	BETA2	.220053	.90 QUANTILE	1.63000
						MAXIMUM	3.07000

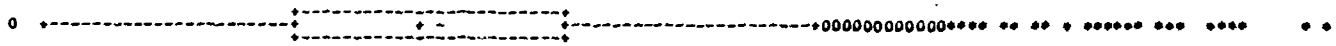
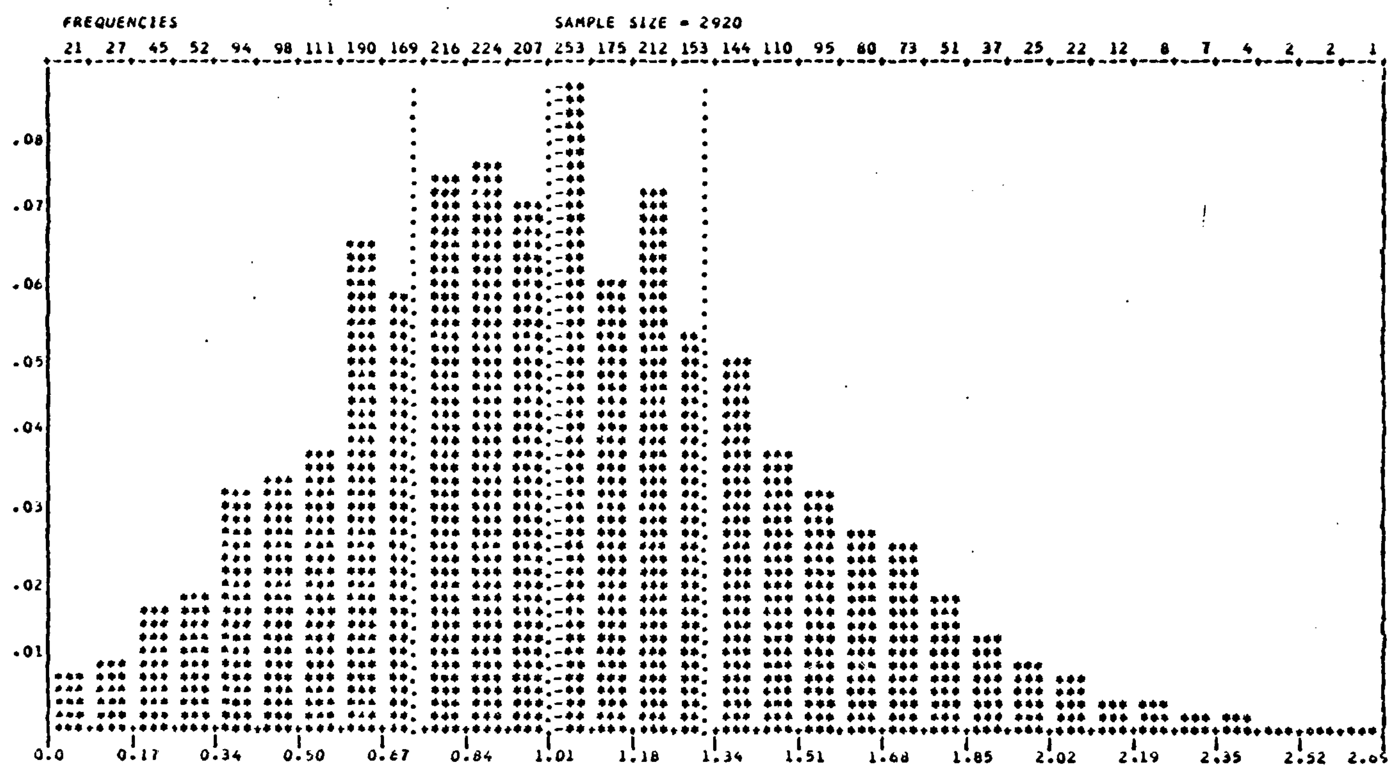


Figure IV.D.1n. Histogram and boxplot of 1968 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.03049	VARIANCE	.196200	M3	.265584E-01	MINIMUM	.0
MEDIAN	1.01000	STD DEV	.442945	M4	.114671	.10 QUANTILE	.470000
TRIMEAN	1.01500	COEF VAR	0.4298	SKEWNESS	0.5056	.25 QUANTILE (HINGE)	.720000
MIDMEAN	1.00899	MEAN DEV	.352205	KURTOSIS	-0.0211	.50 QUANTILE (MEDIAN)	1.01000
MIDRANGE	1.34500	RANGE	2.69000	BETA1	.265311E-01	.75 QUANTILE (HINGE)	1.32000
		MIDSPREAD	.600000	BETA2	.114593	.90 QUANTILE	1.62000
						MAXIMUM	2.65000

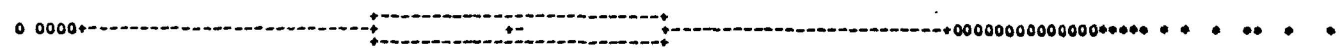
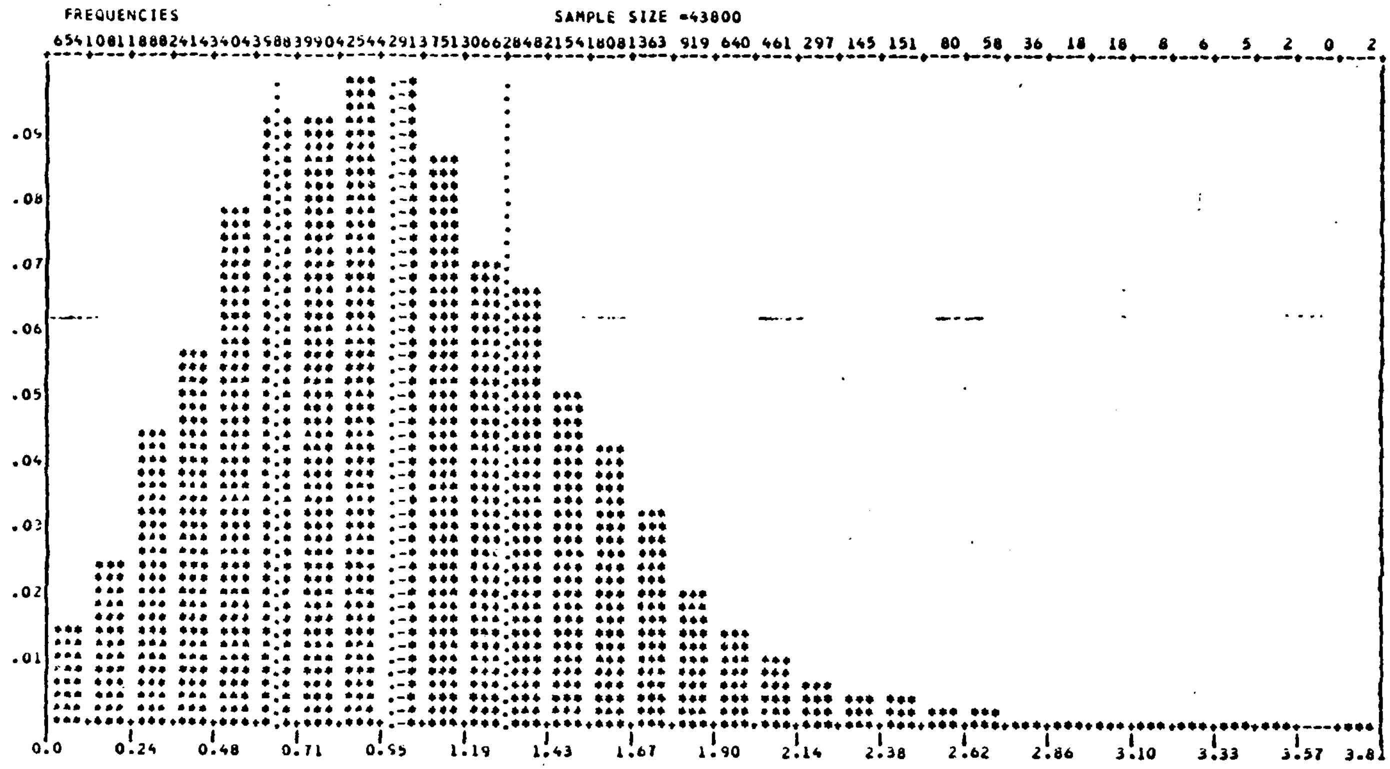


Figure IV.D.lo. Histogram and boxplot of 1969 detrended data.



CENTRAL TENDENCY		SPREAD		HIGHER CENTRAL MOMENTS		DISTRIBUTION	
MEAN	1.02016	VARIANCE	.241845	M3	.641693E-01	MINIMUM	.0
MEDIAN	.960000	STD DEV	.491777	M4	.199959	.10 QUANTILE	.400000
TRIMEAN	.967500	COEF VAR	0.4917	SKEWNESS	0.5395	.25 QUANTILE (HINGE)	.640000
MIDMEAN	.963892	MEAN DEV	.390123	KURTOSIS	0.4187	.50 QUANTILE (MEDIAN)	.960000
MIDRANGE	1.90500	RANGE	3.81000	BETA1	.641649E-01	.75 QUANTILE (HINGE)	1.31000
		MIDSPREAD	.670001	BETA2	.199948	.90 QUANTILE	1.65000
						MAXIMUM	3.81000

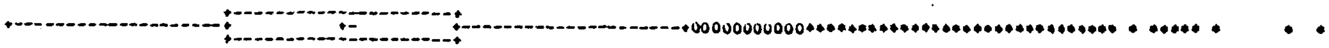


Figure IV.D.1p. Histogram and boxplot of pooled data from years 1955-1969.

Table IV.D.1.

Moment Estimate of the Gamma Shape Parameter by Year

Year	Estimate of k
1955	3.96
1956	4.16
1957	4.38
1958	3.68
1959	3.65
1960	4.45
1961	3.87
1962	3.87
1963	3.86
1964	4.49
1965	4.32
1966	4.04
1967	5.08
1968	4.24
1969	5.41

discretization and clipping that has apparently occurred while the data was collected and processed. As a rough check on the extent of the clipping required to produce values for the kurtosis and skewness similar to those for the data the following procedure was examined. A sample of 2920 values for a Gamma distribution were produced with mean 1 and shape parameter k = 4.0 by a call to the NPS random number generator LLRANDOMII(SGAMA). A histogram of these values was produced sequentially for the following cases. The data was clipped so that all values over four were set equal to four, all values over three were set equal to three, and finally the highest ten percent of the data was set equal to the value of the 2890 sample order statistic. The first four central moments were estimated under each of the conditions. The results are presented in Table IV.D.2. The results indicate that a clipping of the top ten percent of the data will yield results for skewness and kurtosis comparable with those observed in the data.

TABLE IV.D.2

	Mean	SD	CV	Skewness	Kurtosis
Gamma	0.986	0.504	0.511	1.128	2.114
Cut at 4.0	0.986	0.504	0.511	1.128	2.114
Cut at 3.0	0.984	0.498	0.506	0.994	1.165
10% Cut	0.982	0.489	0.498	0.861	0.516

The conclusion that the innovative factors can be modeled as random variables with Gamma marginals when combined with the conclusions from IV.B specify the general form of the model to be possibly that of the GLAR(1) process, although further detrending might indicate that the more general GLARMA(p,q) model of Chapter II might have to be used.

Since the estimated correlations $\rho(k)$ are affected by remaining trend (as seen in Table IV.D.3), it is best to examine the structure of the dependency process via the periodogram.

Figures IV.D.2 through IV.D.5 show the periodogram and log periodogram for the 1955 and 1969 data detrended by the single, yearly harmonic exponential sine (see equation IV.C.2). Superimposed over these plots is the spectral density and log spectral density of a theoretical AR(1) process with correlation equal to 0.849 (see Table IV.D.3). This spectral density is also the spectral density of the GLAR(1) process. We have

$$f(\omega) = \frac{2(1-\rho^2)}{1+\rho^2 - 2\cos(\omega)}, \quad 0 \leq \omega \leq \pi, \qquad \text{(IV.D.2)}$$

with $\rho = 0.849$.

All of these plots show that the detrending has reduced the importance of the yearly cycle and that a six month cycle has now become the dominant factor. The theoretical GLAR(1) spectral density fits well for the periodogram after the point representing the six month cycle. The six month cycle

Table IV.D.3. High values for average data shows that a remaining trend is inflating these correlations (see Figure IV.D.7.).

DATA: 3-HOURLY WIND VELOCITIES; DETRENDING: 1 HARMONIC EXPONENTIAL SINE

LAG	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	15 YR AVG	15 YR SD	AVG DATA
1	0.82	0.80	0.81	0.82	0.83	0.80	0.82	0.82	0.84	0.80	0.85	0.84	0.82	0.88	0.85	0.826	0.020	0.849
2	0.69	0.66	0.67	0.69	0.71	0.67	0.68	0.70	0.71	0.66	0.71	0.70	0.69	0.76	0.71	0.695	0.024	0.745
3	0.56	0.54	0.54	0.58	0.59	0.56	0.55	0.58	0.59	0.54	0.59	0.58	0.56	0.64	0.58	0.573	0.026	0.645
4	0.45	0.44	0.45	0.50	0.50	0.46	0.45	0.49	0.50	0.45	0.49	0.50	0.48	0.53	0.48	0.478	0.025	0.584
5	0.36	0.36	0.36	0.42	0.40	0.37	0.35	0.41	0.41	0.35	0.40	0.41	0.39	0.44	0.38	0.387	0.028	0.504
6	0.28	0.30	0.30	0.37	0.34	0.31	0.27	0.35	0.34	0.28	0.32	0.36	0.34	0.36	0.31	0.324	0.031	0.464
7	0.22	0.25	0.25	0.34	0.29	0.27	0.20	0.30	0.30	0.21	0.27	0.30	0.29	0.30	0.26	0.270	0.036	0.424
8	0.18	0.22	0.24	0.31	0.26	0.23	0.15	0.27	0.27	0.19	0.23	0.26	0.26	0.27	0.23	0.238	0.040	0.410
9	0.14	0.18	0.19	0.28	0.23	0.20	0.11	0.23	0.24	0.15	0.19	0.20	0.20	0.24	0.20	0.198	0.041	0.364
10	0.12	0.16	0.17	0.24	0.20	0.17	0.08	0.20	0.20	0.11	0.17	0.18	0.18	0.21	0.20	0.173	0.040	0.344
11	0.12	0.15	0.15	0.22	0.18	0.13	0.06	0.17	0.18	0.08	0.15	0.14	0.13	0.20	0.19	0.152	0.041	0.315
12	0.13	0.16	0.16	0.21	0.17	0.11	0.06	0.17	0.18	0.08	0.14	0.13	0.12	0.19	0.19	0.147	0.041	0.315
13	0.13	0.15	0.15	0.21	0.16	0.08	0.05	0.17	0.17	0.07	0.12	0.10	0.08	0.18	0.19	0.134	0.046	0.292
14	0.13	0.15	0.15	0.20	0.16	0.07	0.05	0.17	0.15	0.06	0.12	0.08	0.07	0.17	0.19	0.128	0.047	0.250
15	0.12	0.14	0.13	0.20	0.15	0.06	0.05	0.16	0.14	0.05	0.11	0.06	0.05	0.18	0.18	0.121	0.051	0.278
16	0.12	0.16	0.14	0.20	0.16	0.06	0.06	0.16	0.14	0.07	0.11	0.06	0.05	0.18	0.18	0.122	0.049	0.284
17	0.10	0.14	0.13	0.18	0.14	0.04	0.04	0.15	0.12	0.06	0.10	0.03	0.04	0.18	0.17	0.108	0.051	0.251
18	0.09	0.15	0.13	0.17	0.14	0.03	0.04	0.14	0.11	0.05	0.11	0.03	0.04	0.18	0.15	0.103	0.051	0.234
19	0.08	0.14	0.12	0.16	0.14	0.02	0.03	0.14	0.11	0.04	0.12	0.02	0.03	0.17	0.14	0.096	0.052	0.213
20	0.07	0.13	0.13	0.15	0.13	0.02	0.04	0.13	0.11	0.04	0.13	0.02	0.04	0.17	0.11	0.095	0.049	0.222
21	0.06	0.12	0.12	0.14	0.12	0.01	0.02	0.11	0.11	0.03	0.13	0.02	0.04	0.17	0.09	0.086	0.049	0.200
22	0.06	0.13	0.13	0.13	0.12	0.02	0.02	0.09	0.11	0.04	0.13	0.02	0.07	0.16	0.08	0.086	0.047	0.207
23	0.06	0.12	0.12	0.13	0.11	0.03	0.01	0.07	0.13	0.04	0.13	0.02	0.07	0.14	0.07	0.083	0.044	0.201
24	0.06	0.13	0.12	0.13	0.11	0.03	0.02	0.07	0.13	0.04	0.14	0.03	0.09	0.13	0.05	0.085	0.042	0.223
25	0.06	0.12	0.10	0.12	0.10	0.02	0.01	0.05	0.13	0.02	0.14	0.03	0.07	0.12	0.04	0.074	0.043	0.215
26	0.05	0.12	0.10	0.10	0.10	0.02	0.01	0.04	0.12	0.02	0.13	0.03	0.06	0.10	0.02	0.069	0.041	0.221
27	0.04	0.10	0.08	0.10	0.09	0.04	0.01	0.04	0.11	0.01	0.12	0.02	0.04	0.09	0.02	0.061	0.039	0.220
28	0.04	0.10	0.08	0.10	0.10	0.04	0.02	0.04	0.10	0.02	0.12	0.03	0.04	0.09	0.02	0.062	0.036	0.245
29	0.04	0.08	0.06	0.08	0.10	0.05	0.02	0.03	0.08	0.02	0.11	0.03	0.02	0.10	0.02	0.055	0.032	0.241
30	0.04	0.08	0.06	0.07	0.10	0.05	0.01	0.03	0.07	0.01	0.10	0.04	0.02	0.11	0.02	0.055	0.032	0.259
31	0.05	0.07	0.05	0.08	0.11	0.05	0.01	0.02	0.06	0.01	0.09	0.04	0.01	0.12	0.03	0.054	0.035	0.255
32	0.07	0.09	0.06	0.08	0.12	0.05	0.03	0.05	0.06	0.02	0.09	0.05	0.02	0.15	0.04	0.064	0.034	0.273

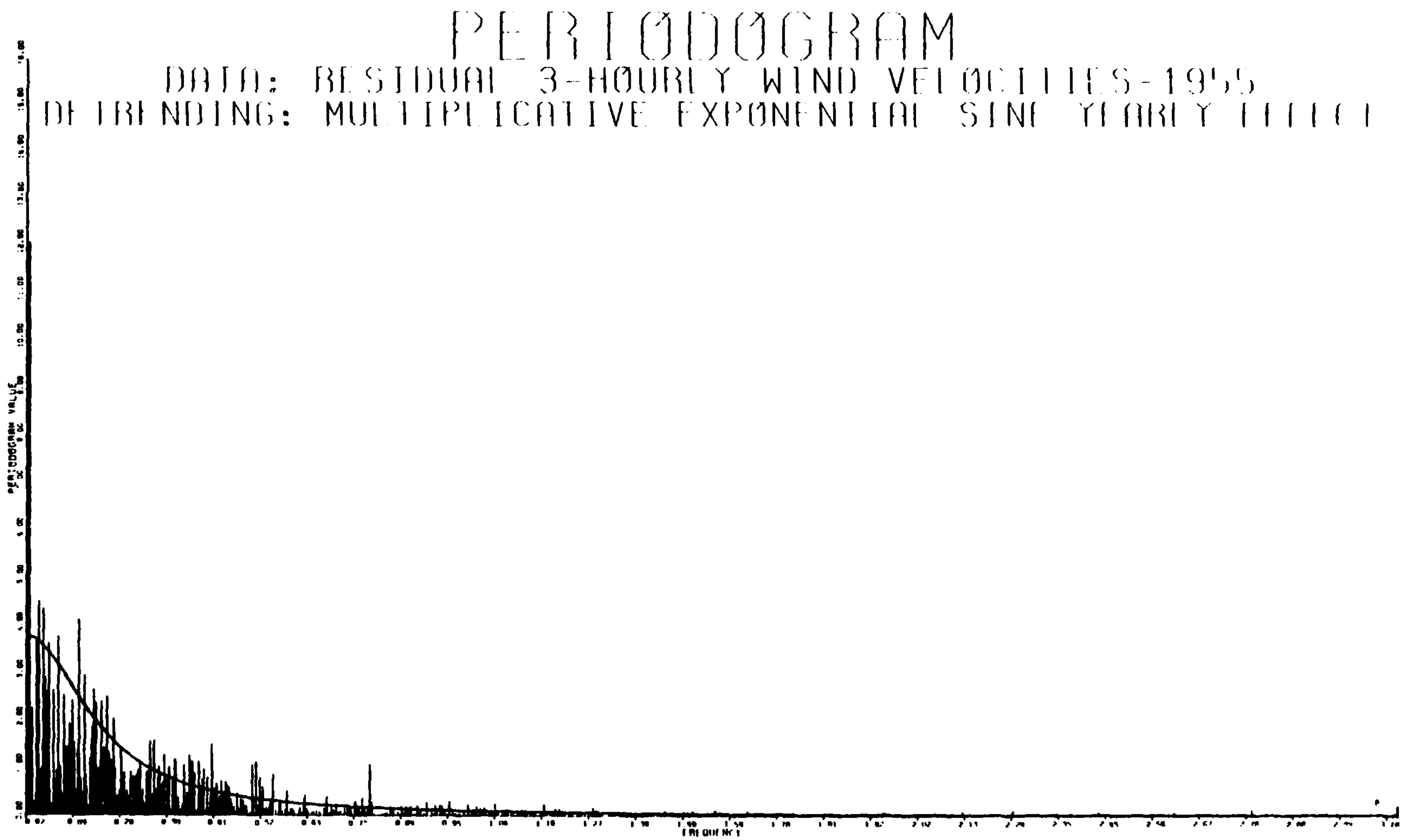


Figure IV.D.2. Spectrum of AR(1) process with correlation of 0.849 is superimposed over periodogram of 1 harmonic detrended data.

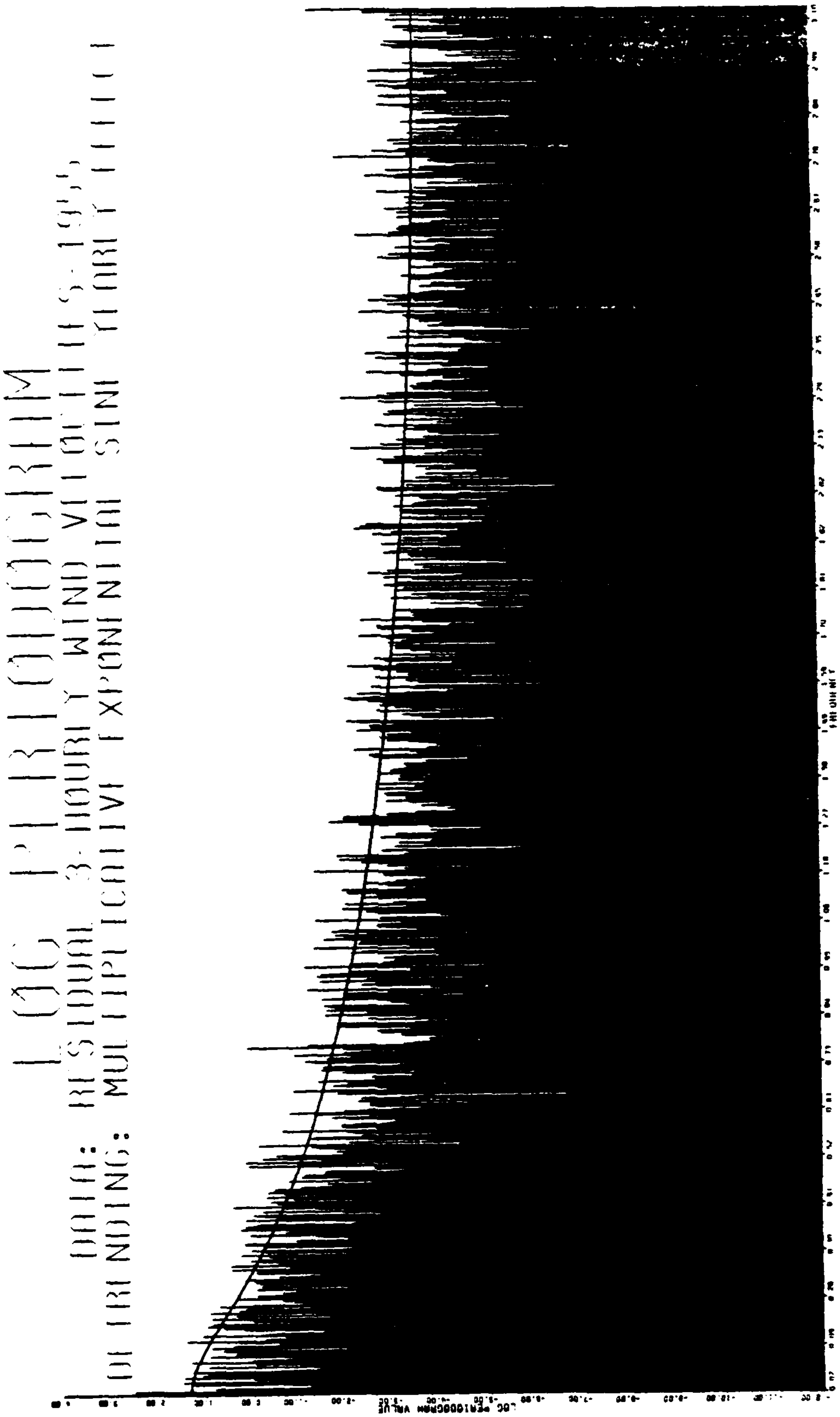

LOG PERIODOGRAM
DATA: RESIDUAL 3-HOURLY WIND VELOCITIES-1955
DETRENDING: MULTIPLICATIVE EXPONENTIAL SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.D.3.

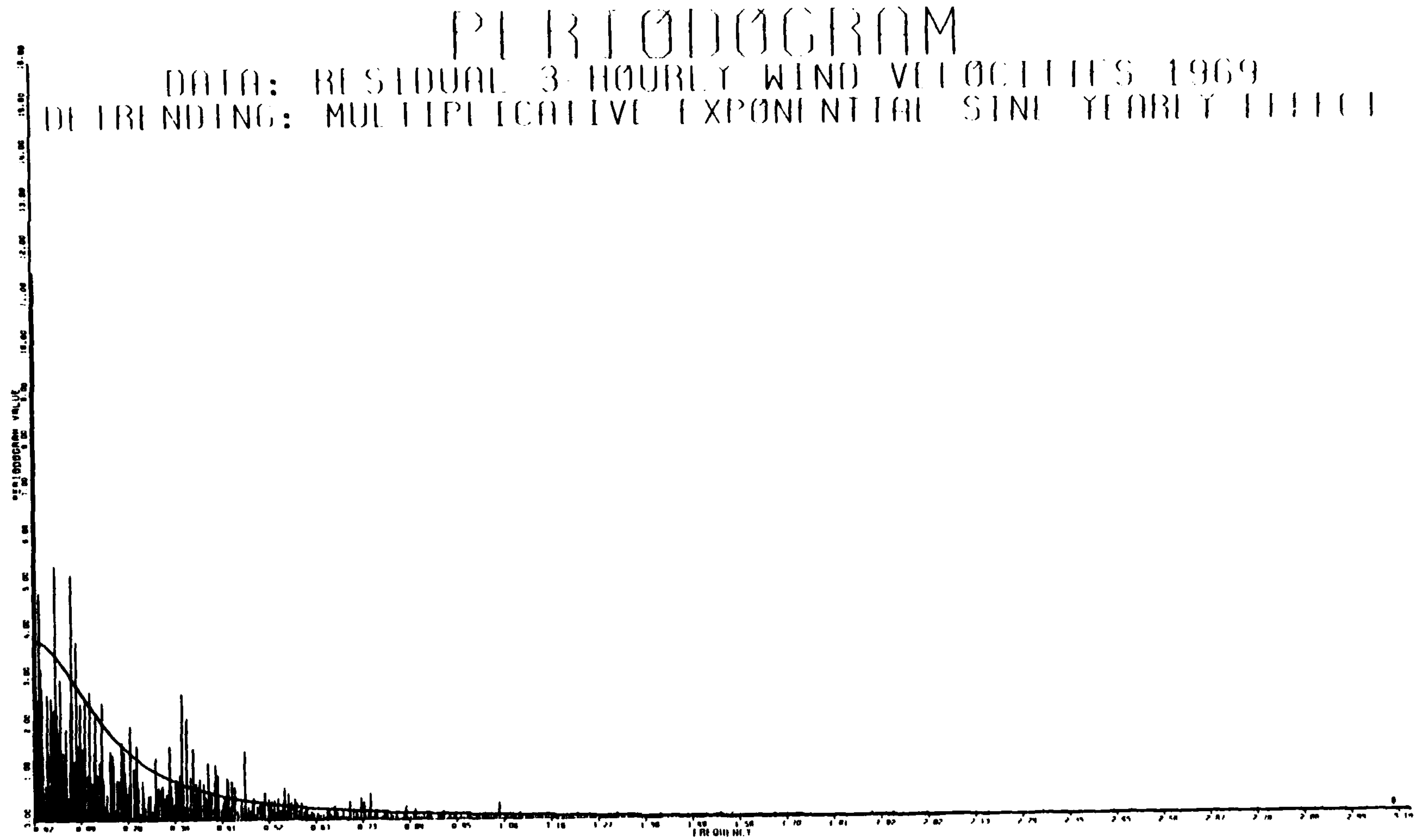


Figure IV.D.4. Spectrum of AR(1) process with correlation of 0.849 is superimposed over periodogram of 1 harmonic detrended data.

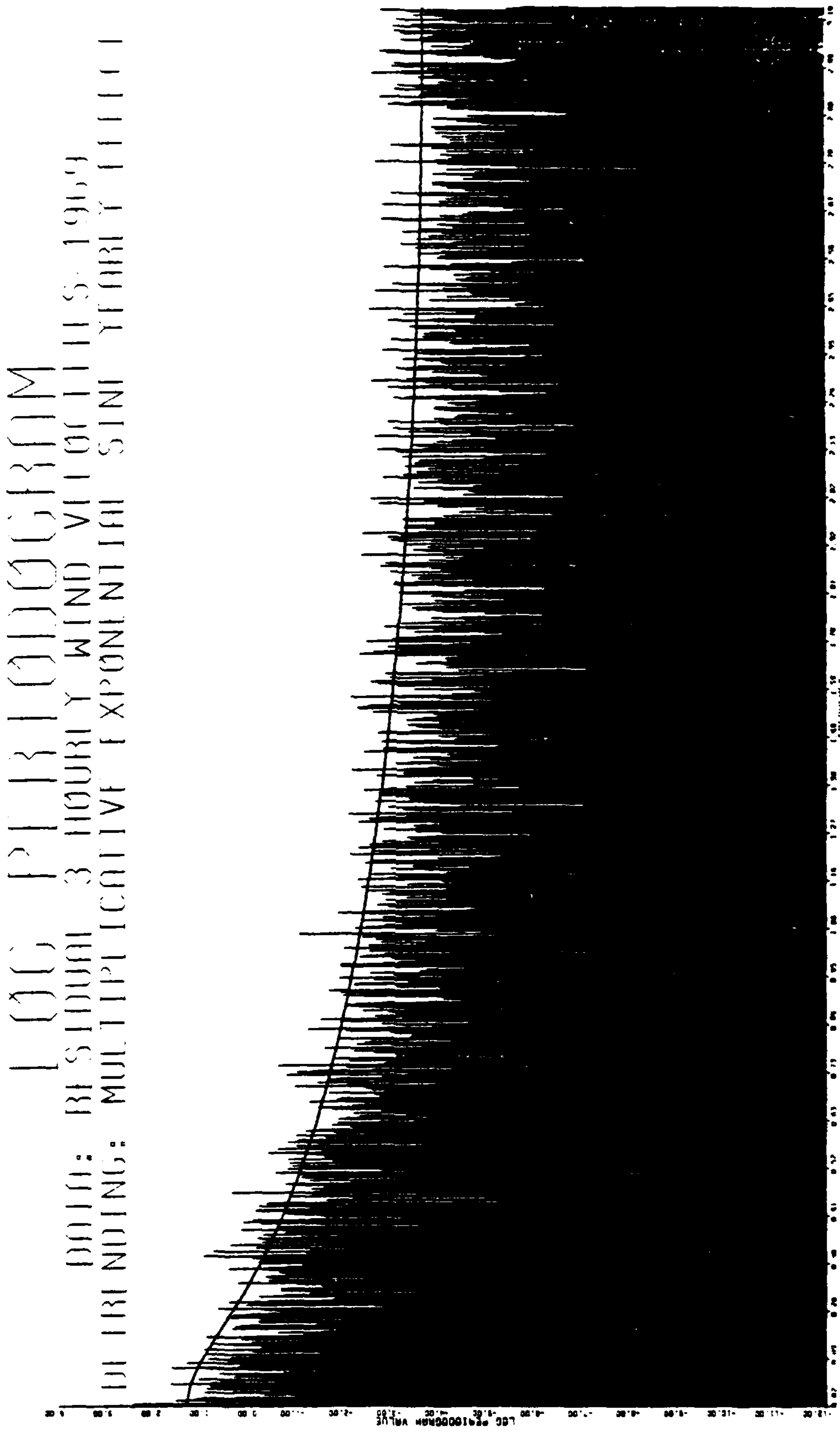

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 1969
DETRENDING: MULTIPLICATIVE EXPONENTIAL SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.D.5.

has now become the dominant factor. Lacking in these plots is an indication of a time of day effect. The appearance of a time of day effect is limited to those plots which use the average data (see IV.C.4 through IV.C.7). Figures IV.C.4 through IV.C.7 and IV.D.2 thorugh IV.D.7 indicate that a further refinement in the model of the mean to include a six month cycle may be helpful. This topic is covered in the next section.

The correlation structure of the detrended data is depicted in Table IV.D.3 and Figures IV.D.6a through IV.D.6p and IV.D.7. Since the one-harmonic year cycle in the data has been reduced, the correlation of the average data in Table IV.D.3 more closely reflects that of the average of the fifteen yearly correlograms. The higher values for the correlations in the average data and its failure to fall below 0.20 may be indications that a trend still exists in the data (the six month trend) which is artificially inflating these values. This may be a further indication of the desirability of including further cycles in the model of the mean. The slight increases in the correlations for lags of 8, 16, and 24 in the average correlogram, Figure IV.D.6p, and the correlogram for the average data, Figure IV.D.7, may indicate a small time of day effect.

E. REFINING THE FORM OF THE MEAN; A FURTHER DETRENDING

Since several plots in the previous section indicated that a six month cycle had become the dominant factor since the removal of the one-harmonic year cycle, a further refinement for the model

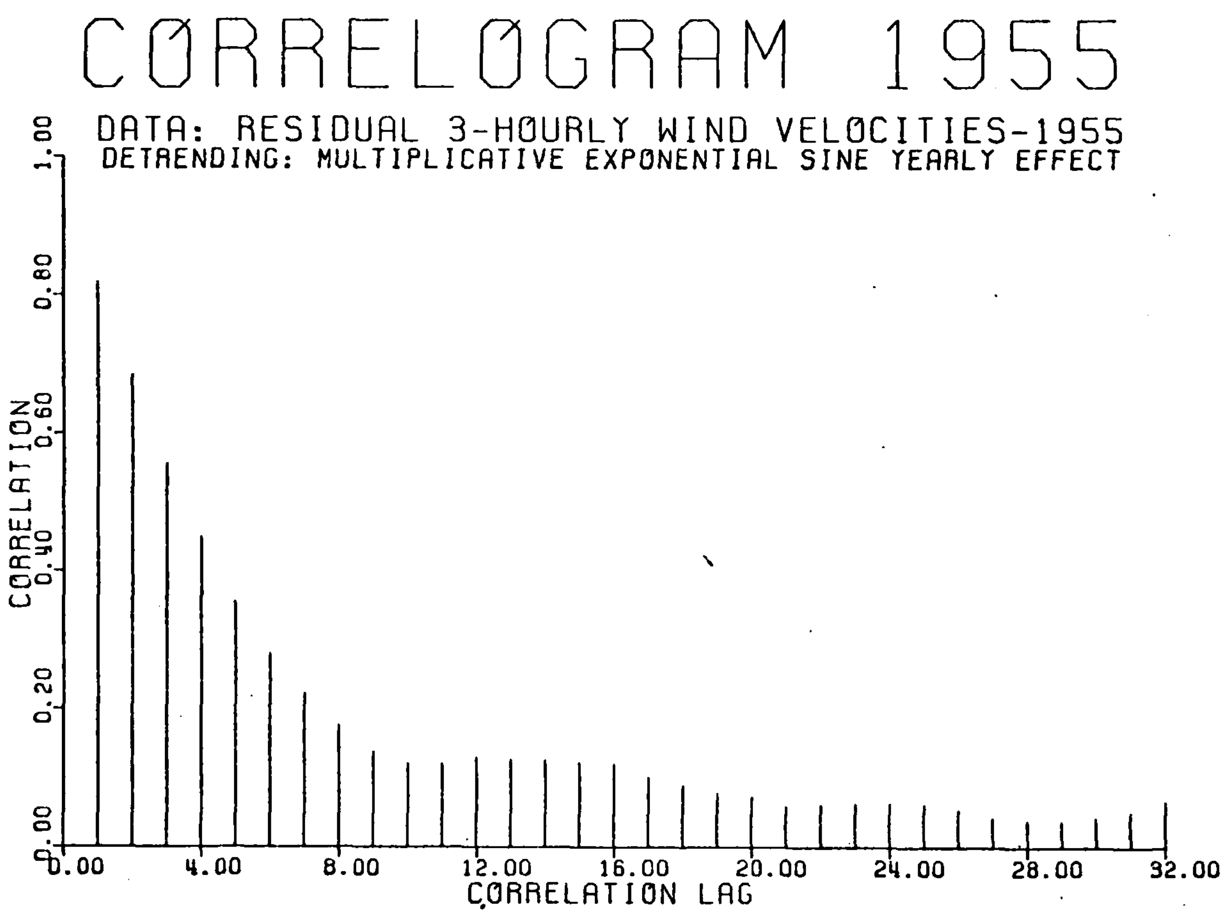


Figure IV.D.6a.

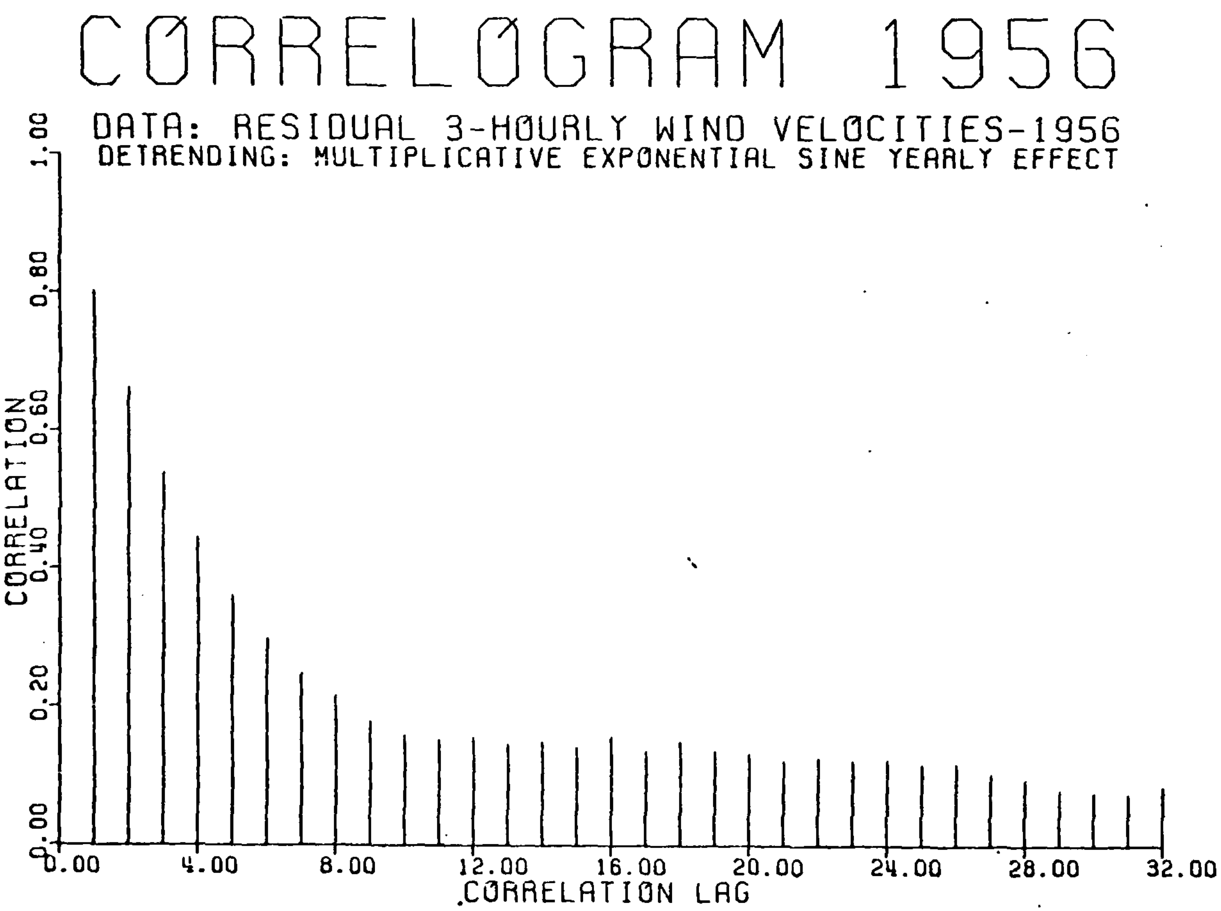


Figure IV.D.6b.

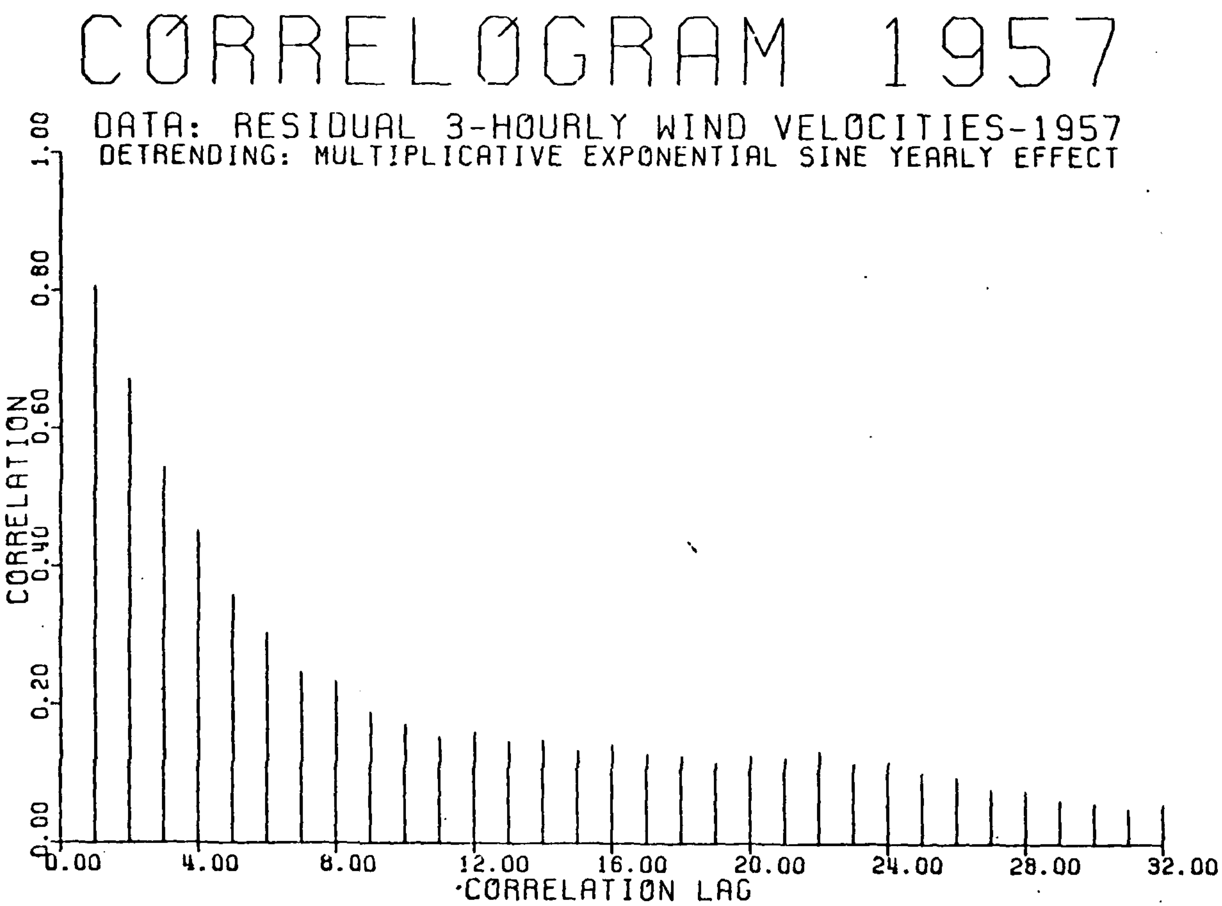


Figure IV.D.6c.

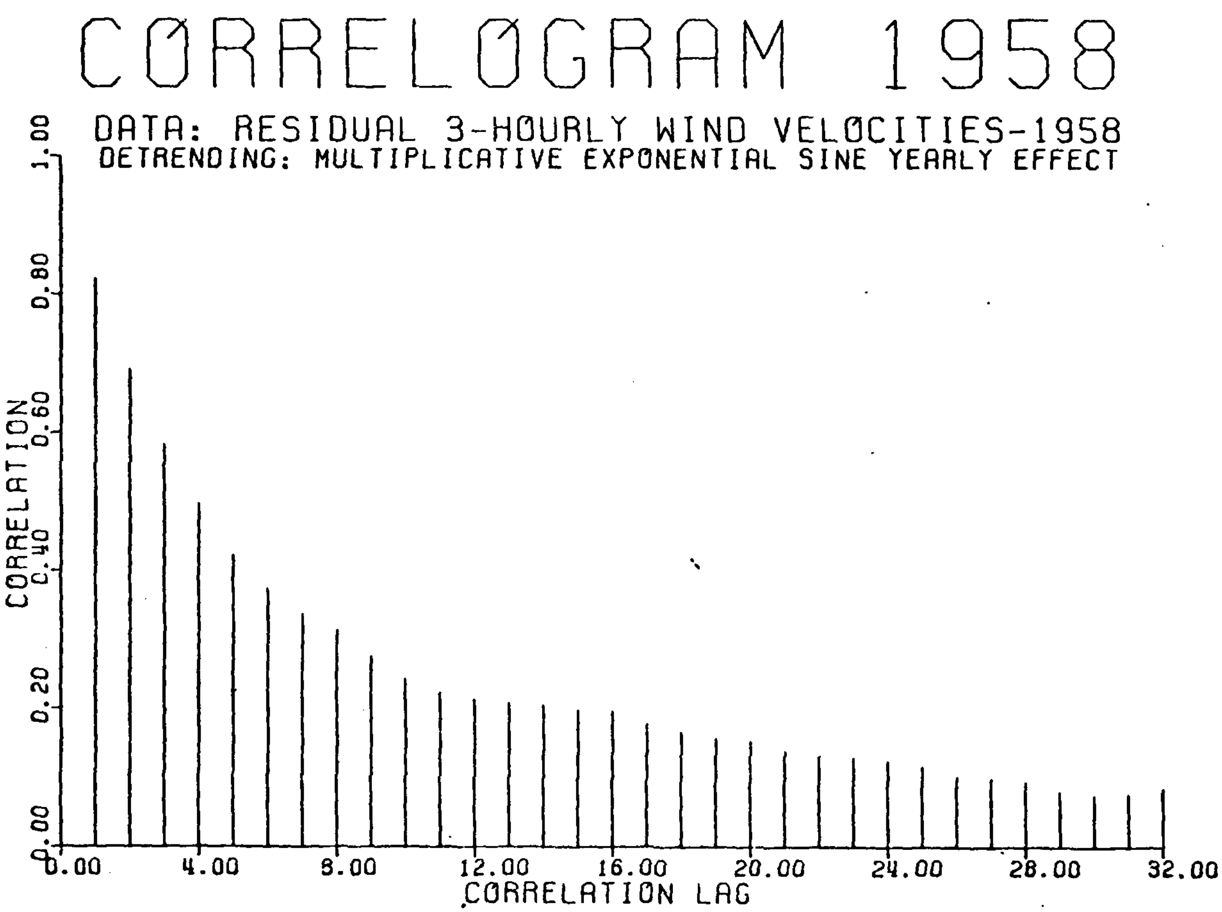


Figure IV.D.6d.

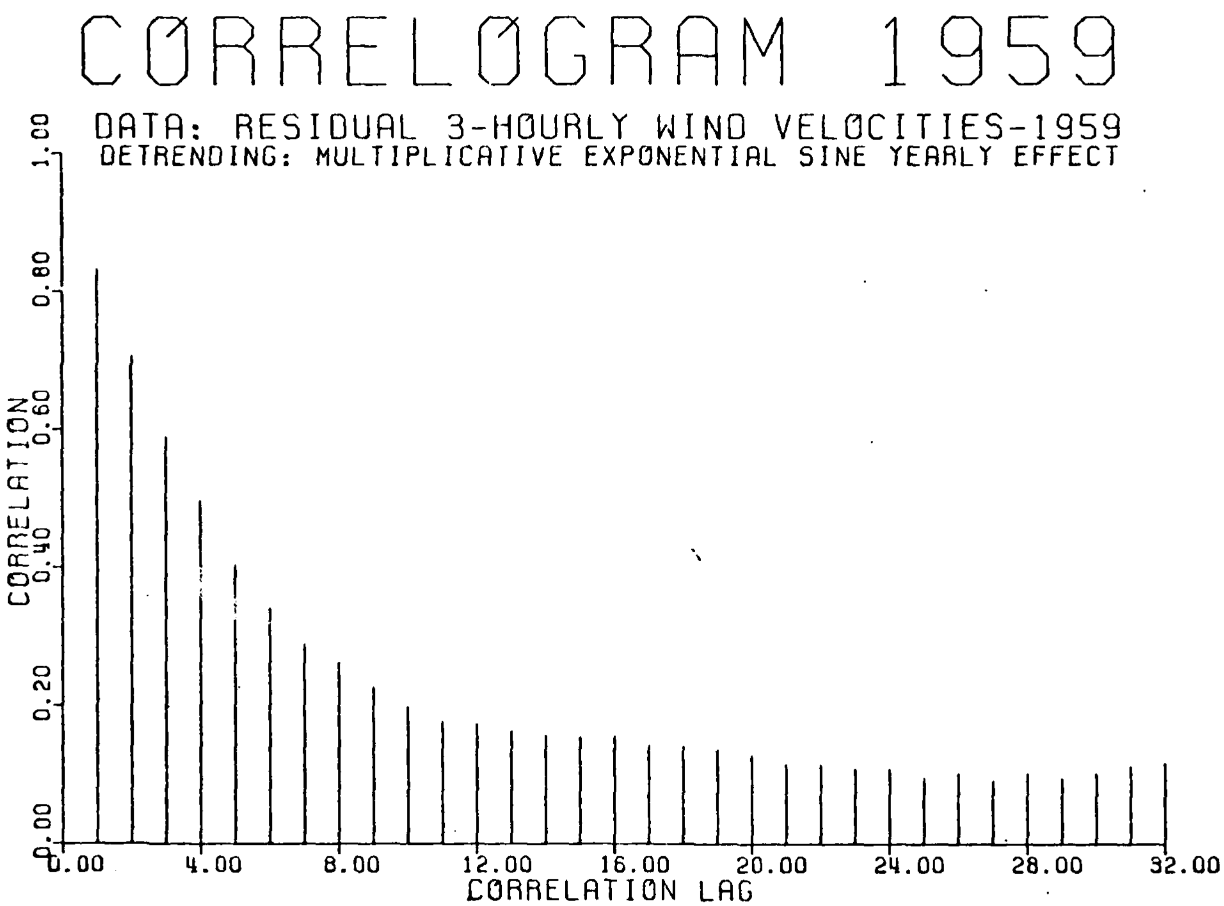


Figure IV.D.6e.

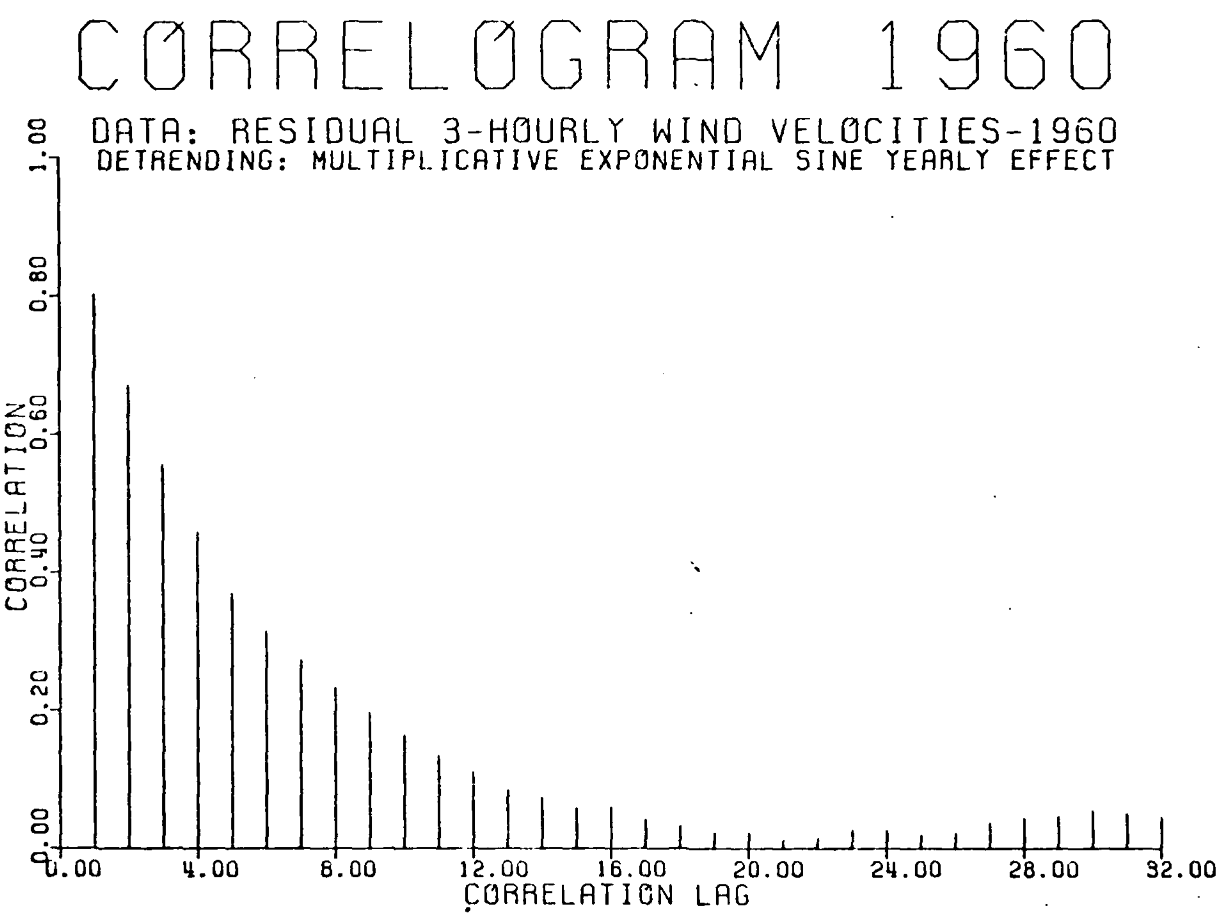


Figure IV.D.6f.

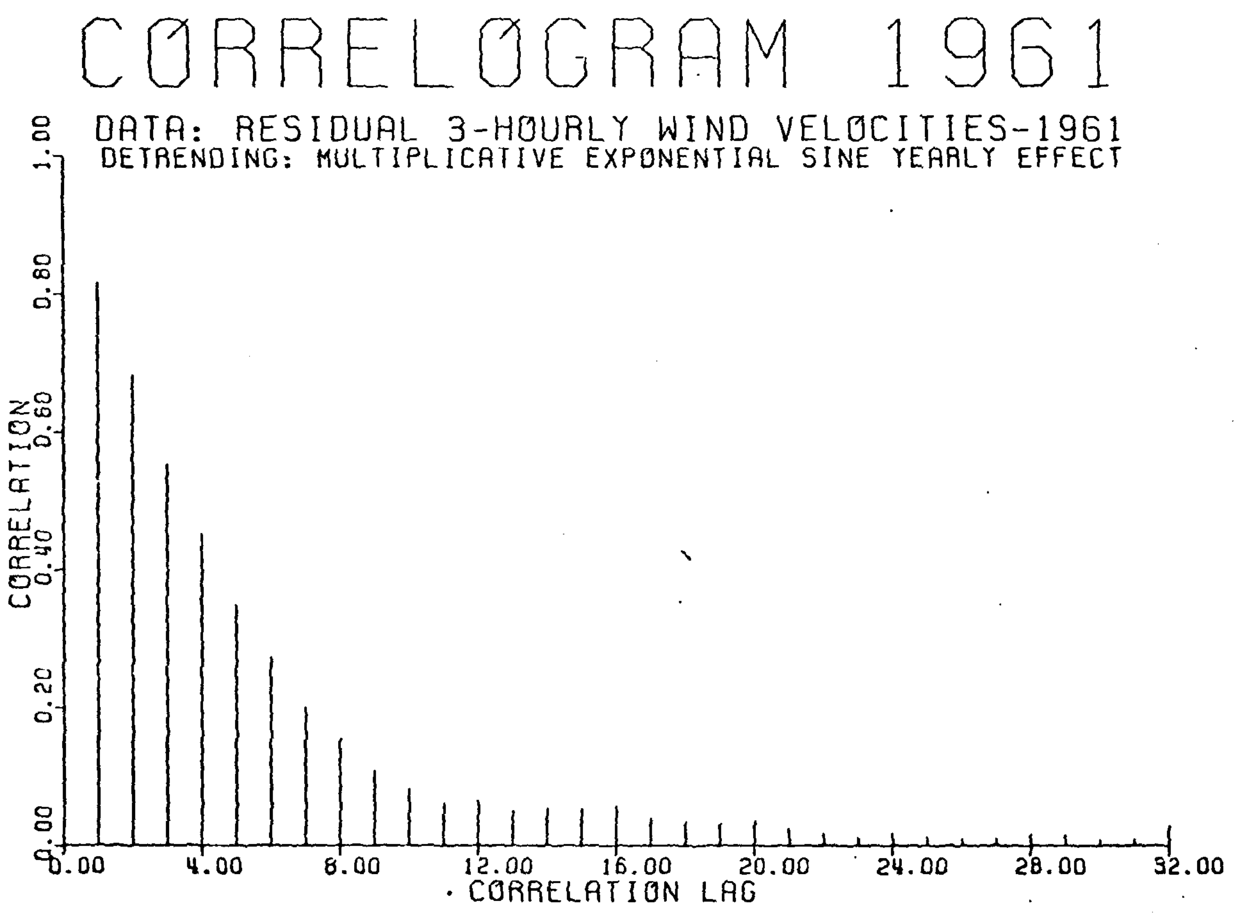


Figure IV.D.6g.

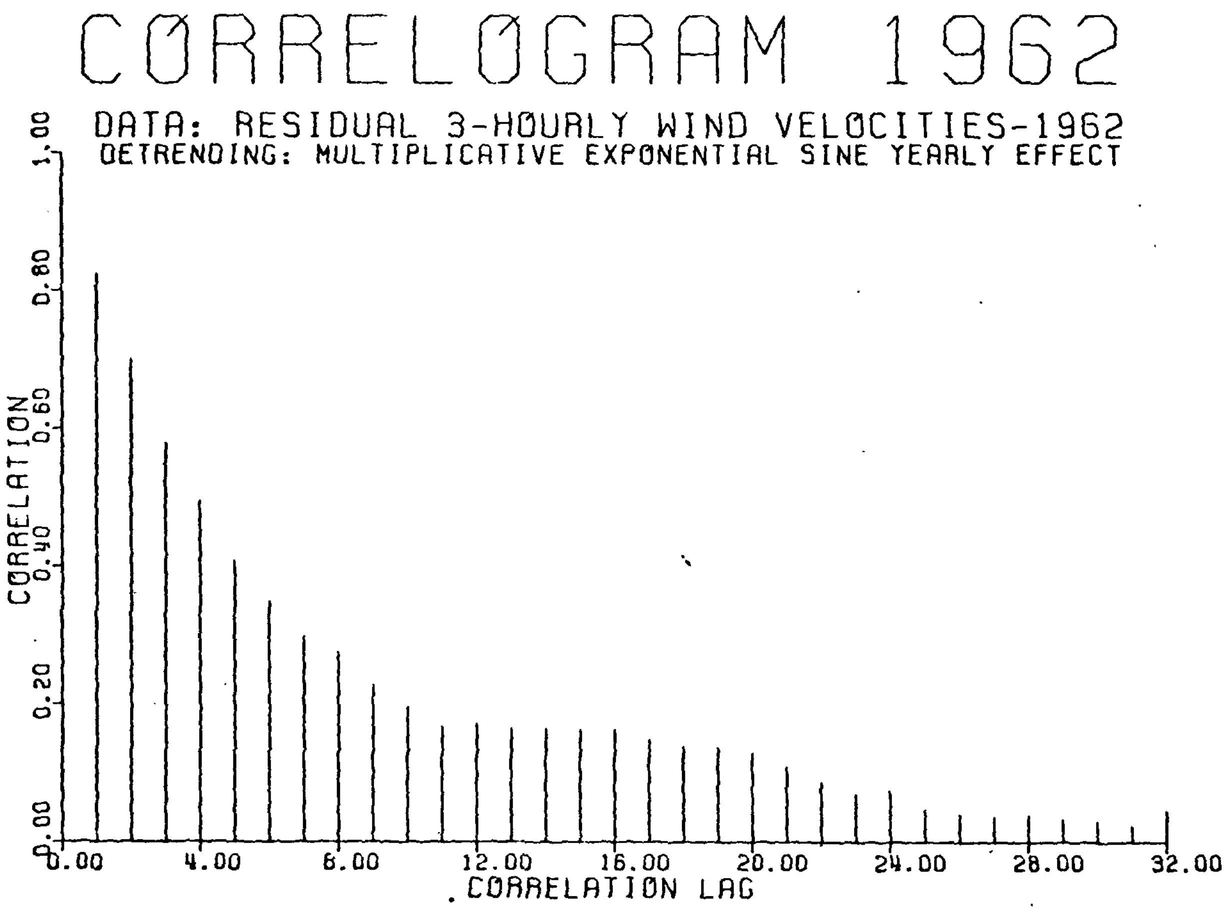


Figure IV.D.6h.

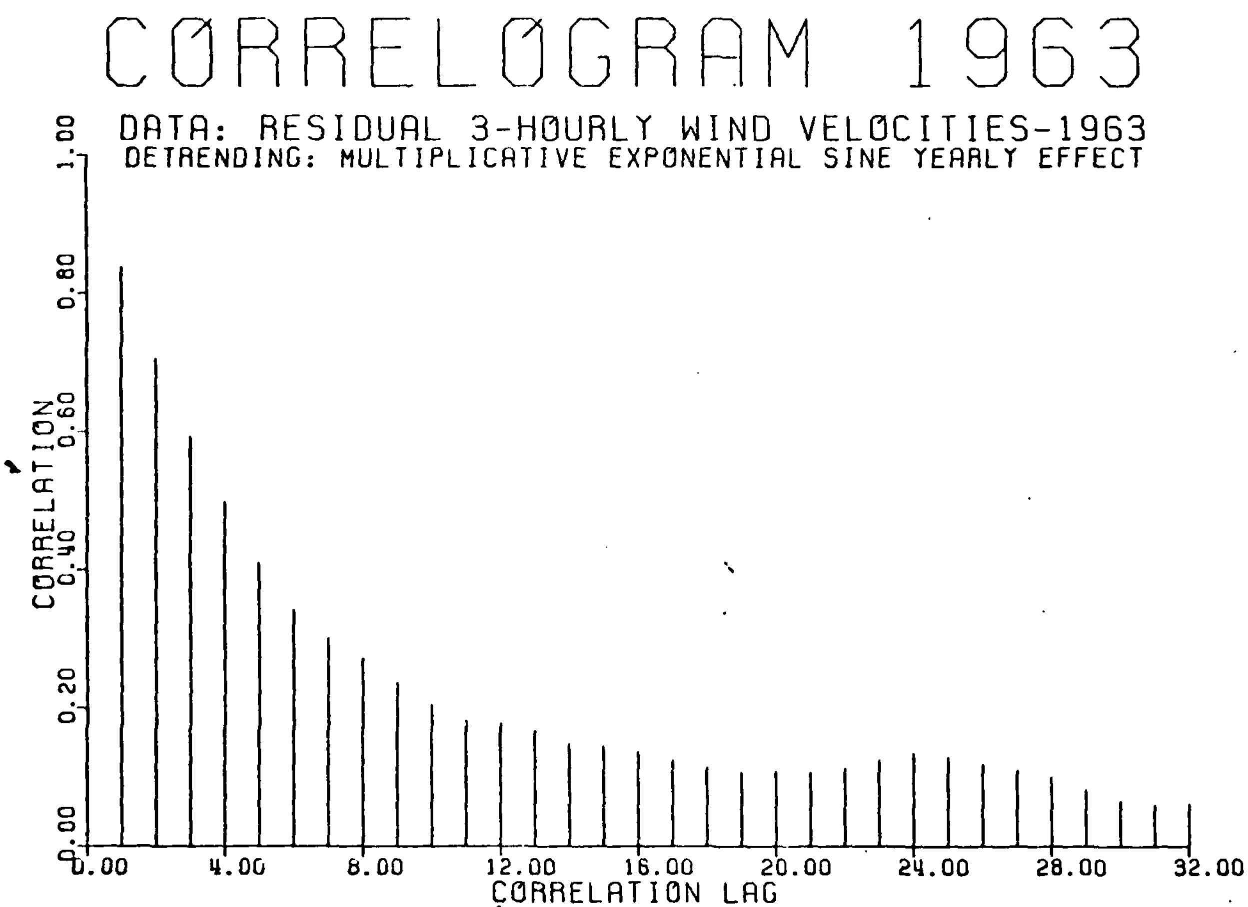


Figure IV.D.6i.

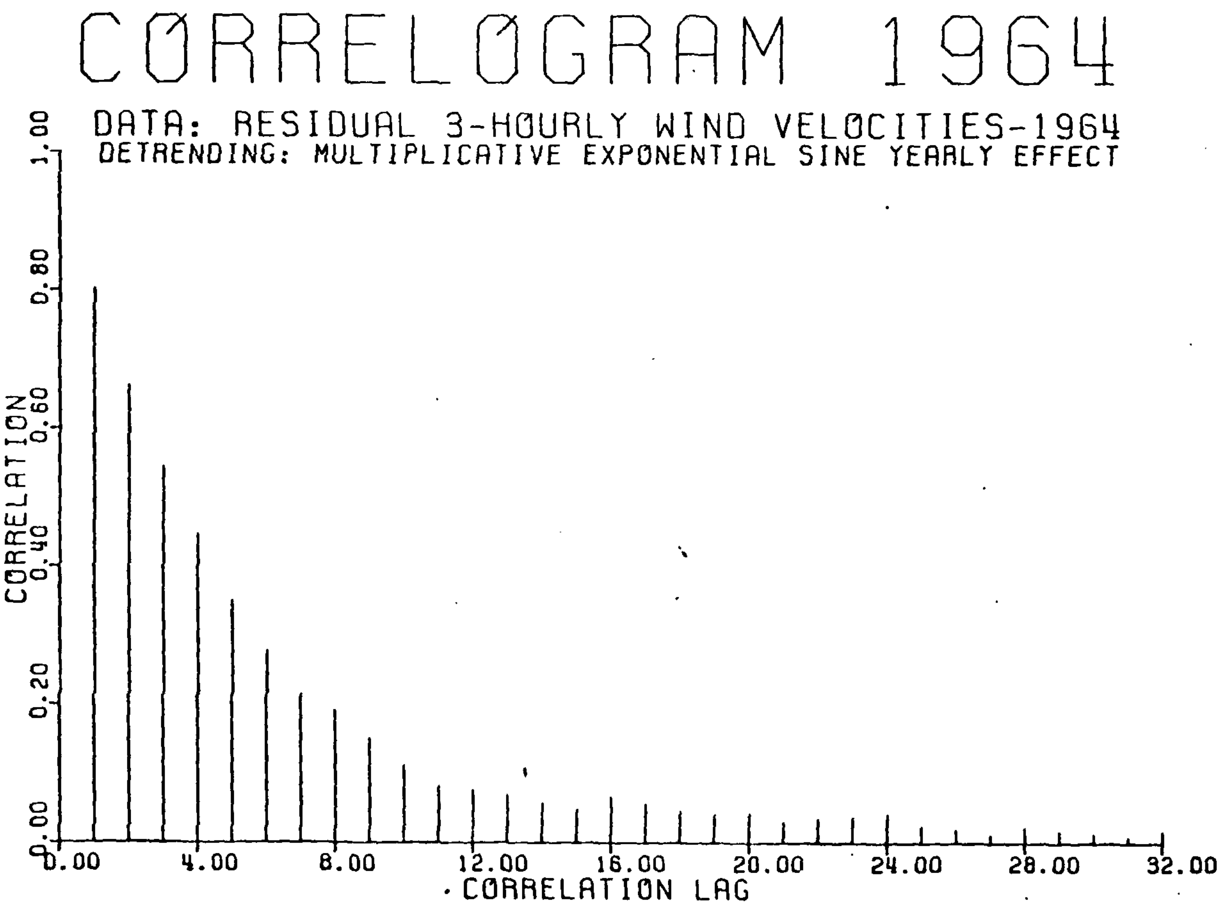


Figure IV.D.6j.

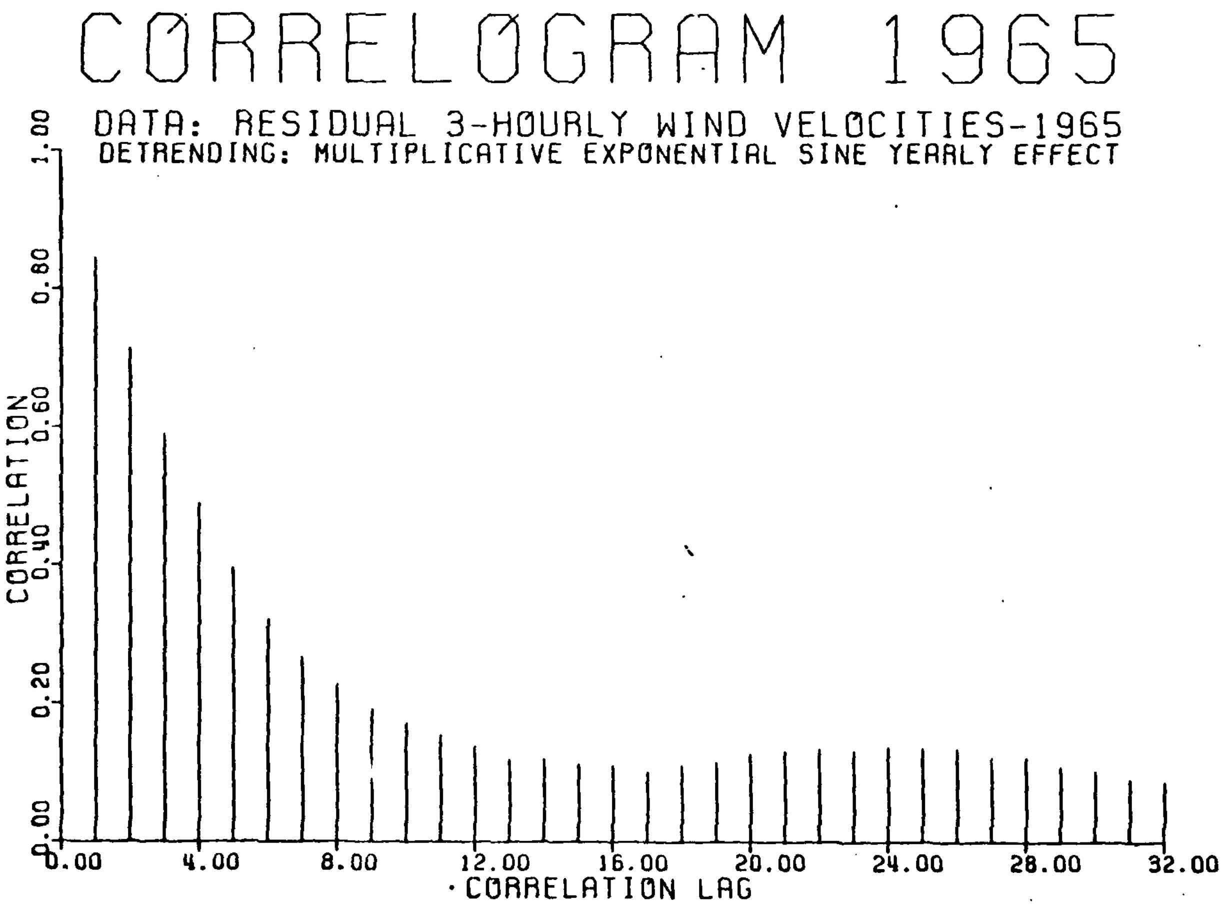


Figure IV.D.6k.

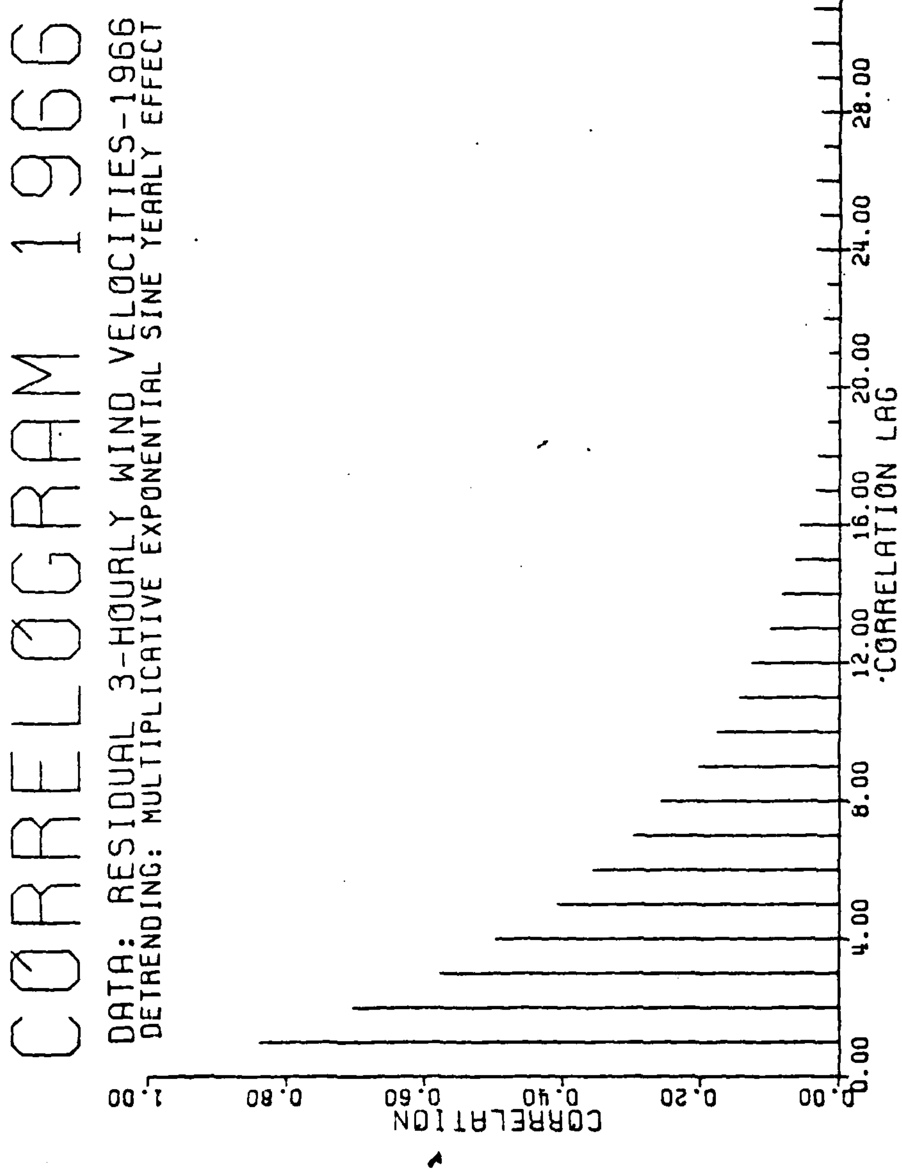


Figure IV.D.61.

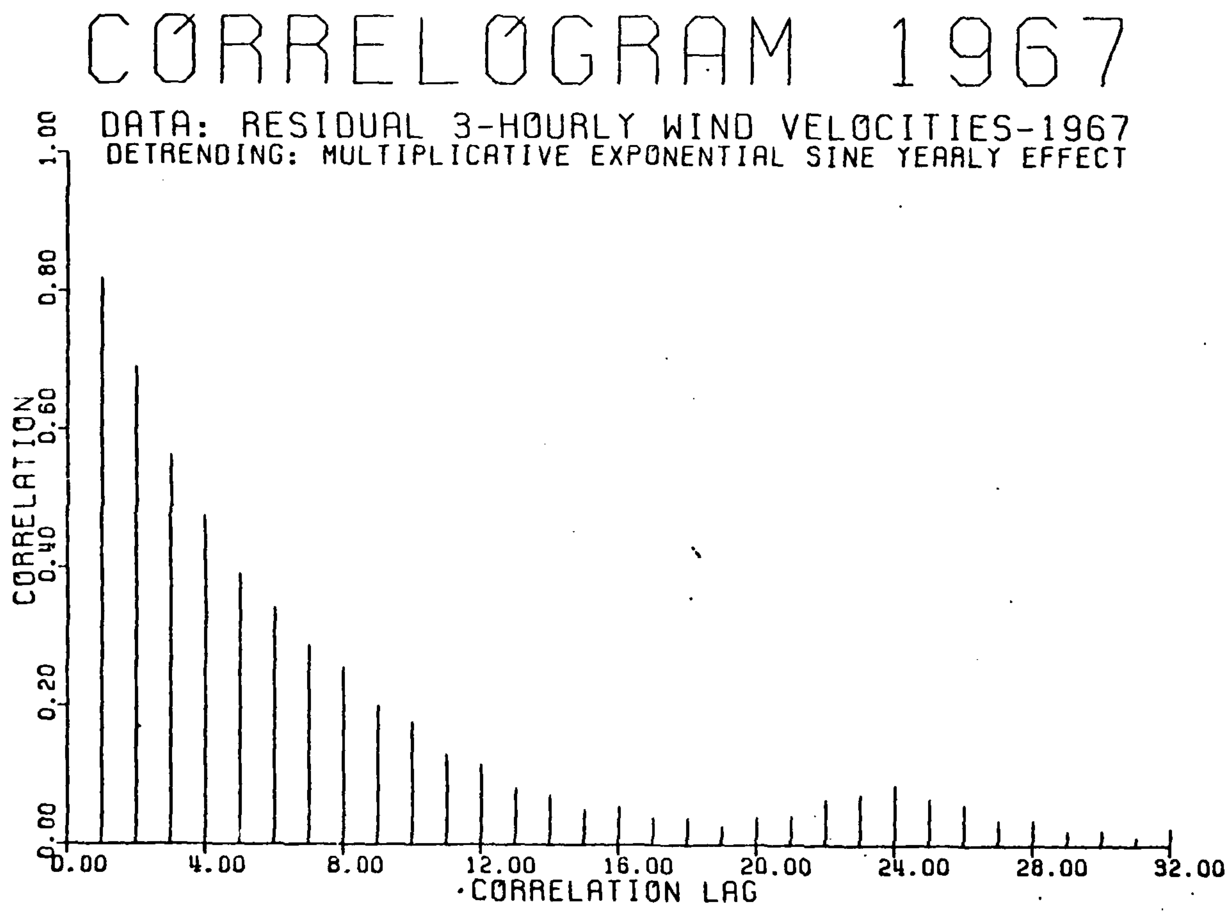


Figure IV.D.6m.

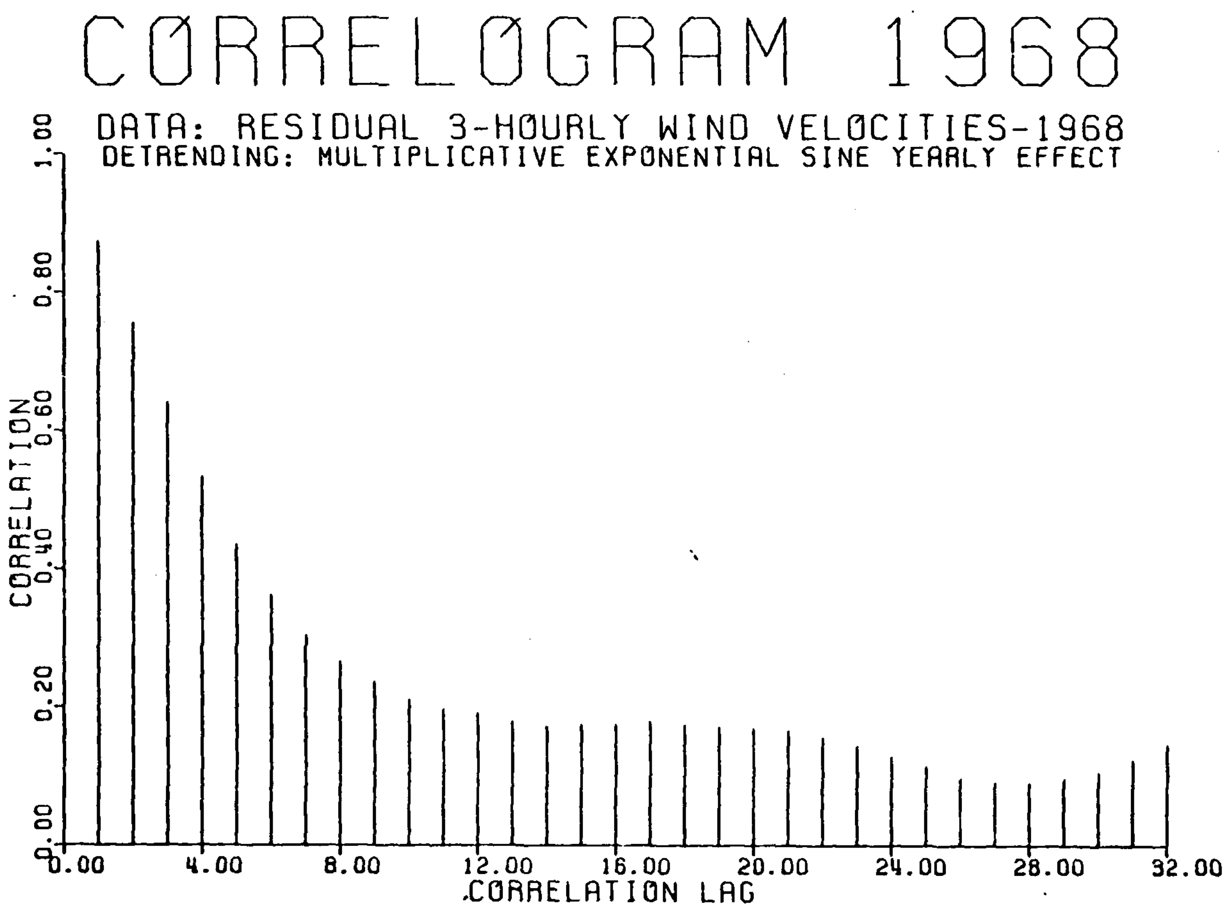


Figure IV.D.6n.

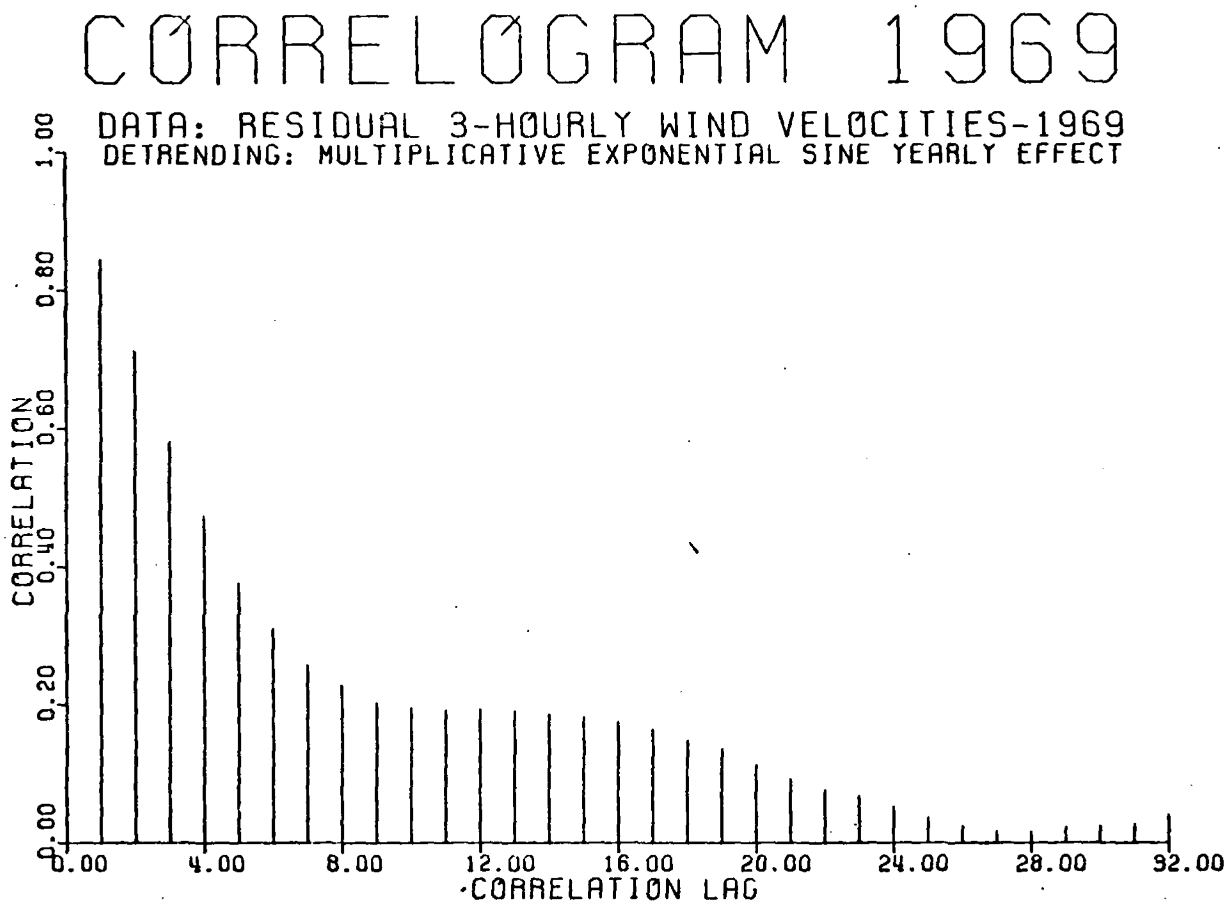


Figure IV.D.6o.

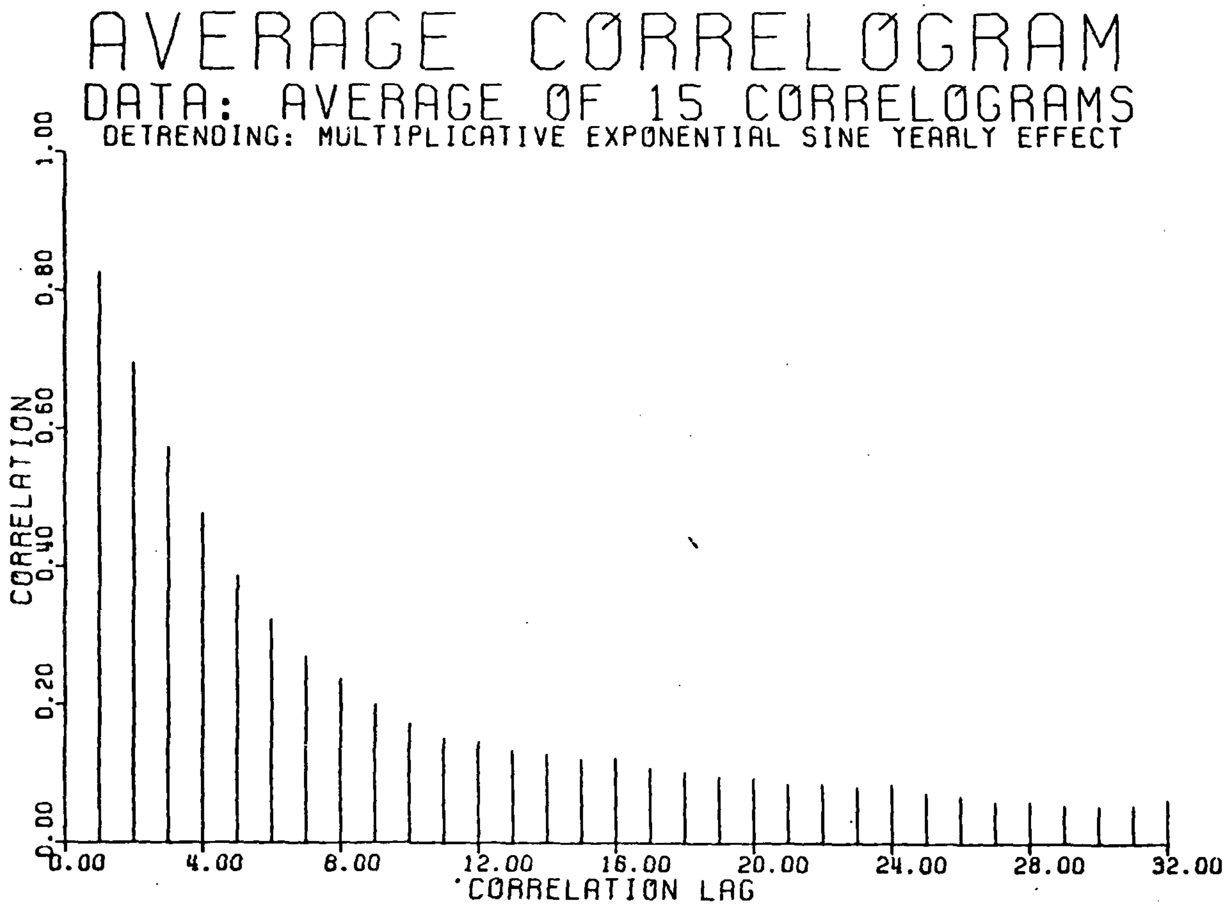


Figure IV.D.6p.

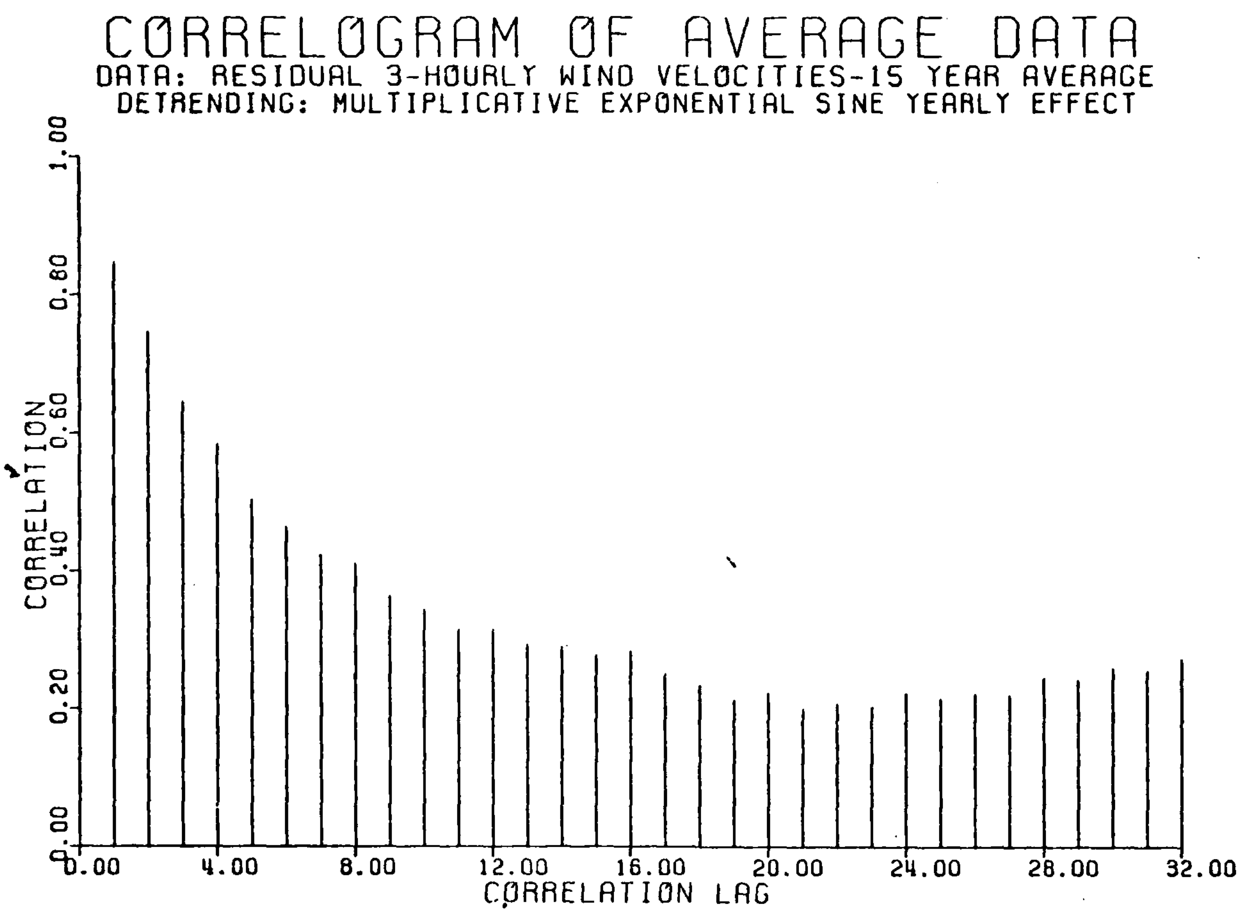


Figure IV.D.7. Failure of correlations to show geometric decline indicates a remaining trend which is artifically inflating the correlation.

of the mean was considered. In this refinement, the sinusoidal model for the mean was reintroduced to the analysis to see if the addition of the second cycle would allow this model to outperform the exponential sine. With the two cycles included the sinusoidal model becomes

$$\mu_n = a + b_1 \sin\left(\frac{2\pi n}{2920}\right) + b_2\left(\frac{2\pi n}{2920}\right) + b_3 \sin\left(\frac{2\pi n}{1460}\right) + b_4 \cos\left(\frac{2\pi n}{1460}\right). \qquad \text{(IV.E.1)}$$

The exponential sine becomes

$$\mu_n = e^{a+b_1\sin\left(\frac{2\pi n}{2920}\right)+b_2\left(\frac{2\pi n}{2920}\right)+b_3\sin\left(\frac{2\pi n}{1460}\right)+b_4\cos\left(\frac{2\pi n}{1460}\right)}. \qquad \text{(IV.E.2)}$$

Parameters for these models were determined following the procedures in IV.C and the estimated values are listed in Table IV.C.1. The plots of the two resulting values for the mean are presented in Figure IV.E.1. Since the two curves are nearly identical and the exponential sine is preferred on a theoretical basis, the sinusoidal model was not further considered.

Figures IV.E.2 through IV.E.5 show the periodogram and log periodogram for 1955 and 1969 after detrending with the two harmonic exponential sine. The value for an AR(1) process is superimposed as before with a correlation of 0.794

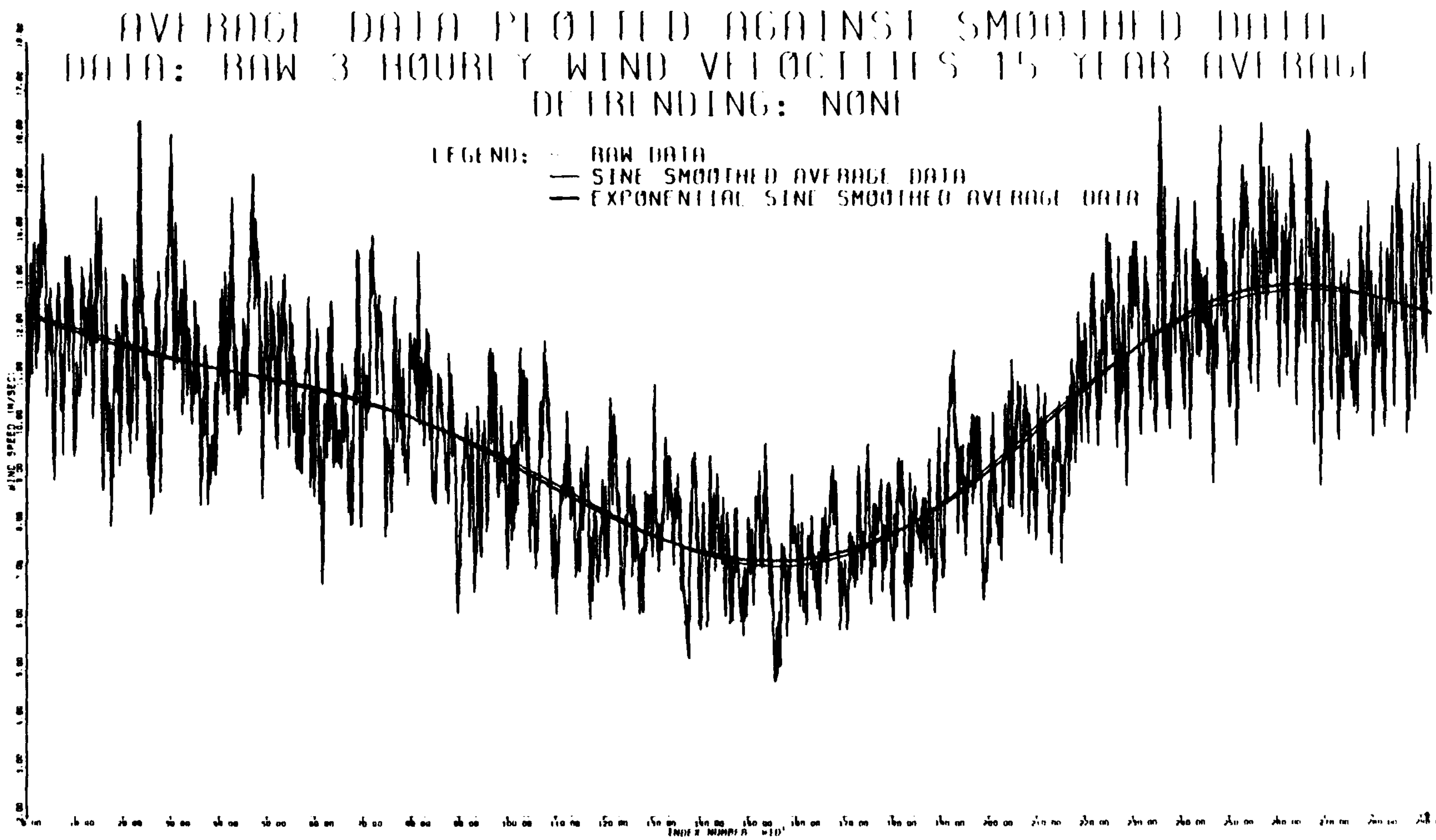


Figure IV.E.1. With 2 harmonics included the fit is much improved (see Figure IV.C.1) and the results of the two models are comparable.

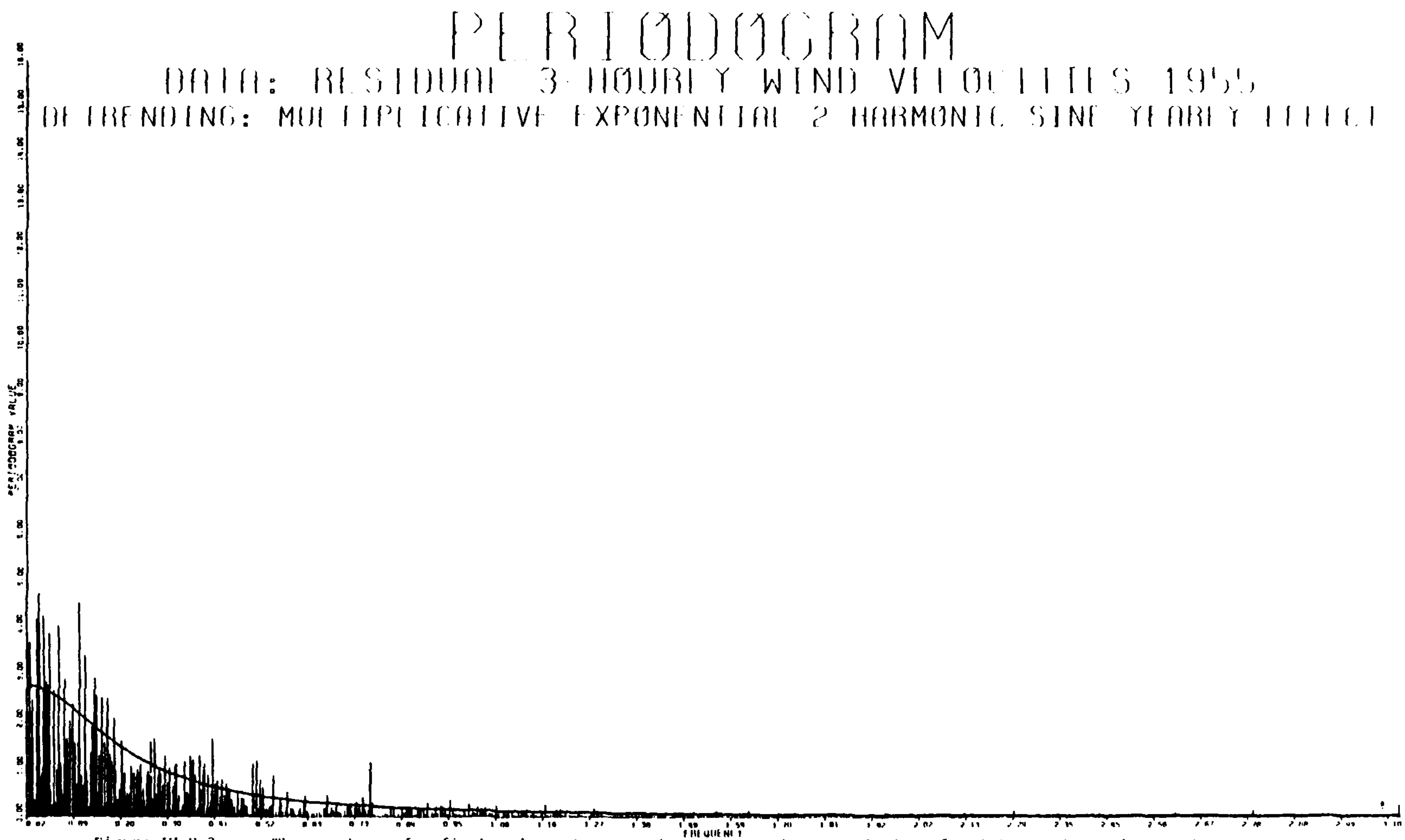


Figure IV.E.2. The spectrum of a first-order autoregressive process with correlation of 0.794 superimposed over the periodogram of the 2 harmonic detrended data. Time of the cycles are not evident.

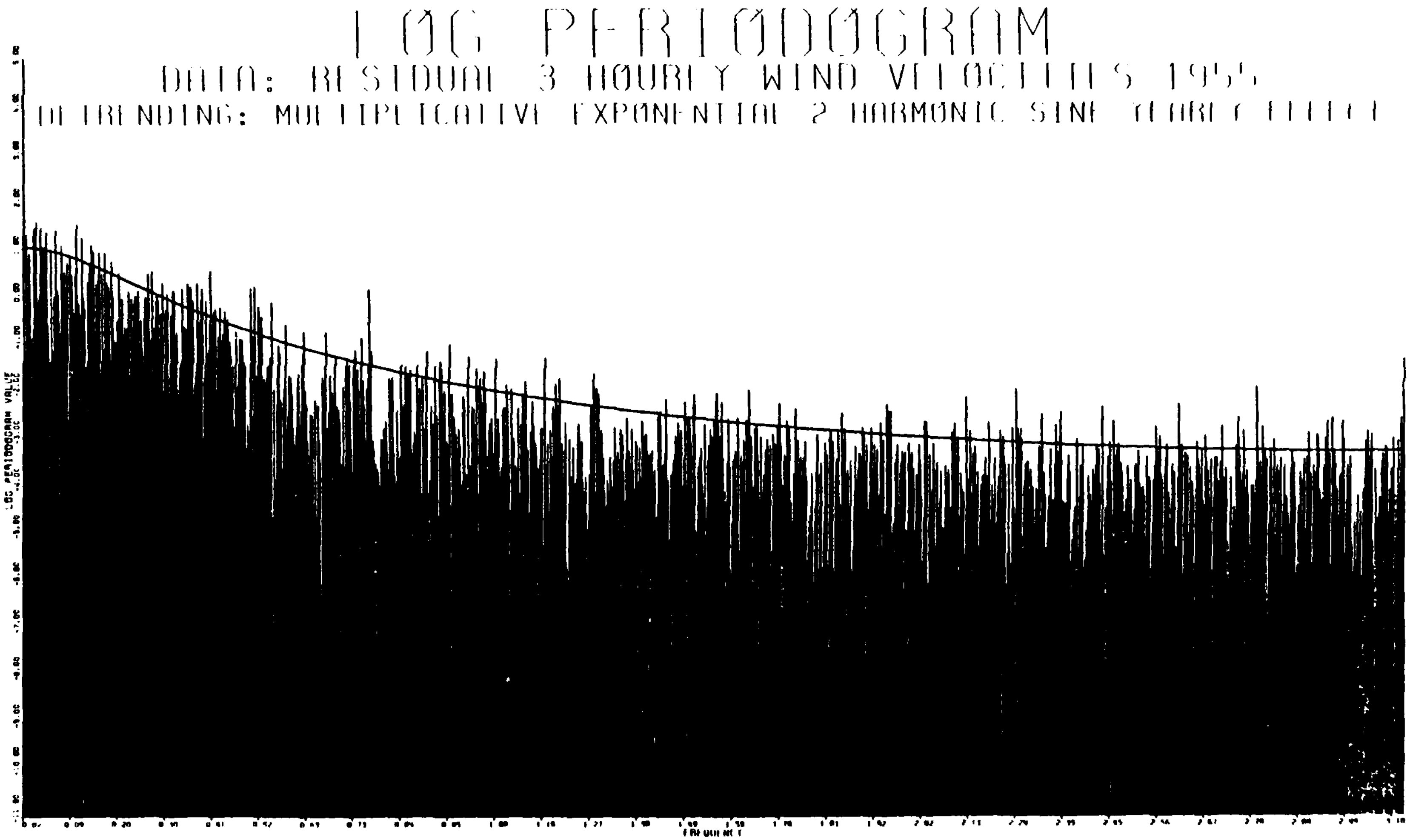

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES
DETRENDING: MULTIPLICATIVE EXPONENTIAL 2 HARMONIC SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.E.3.

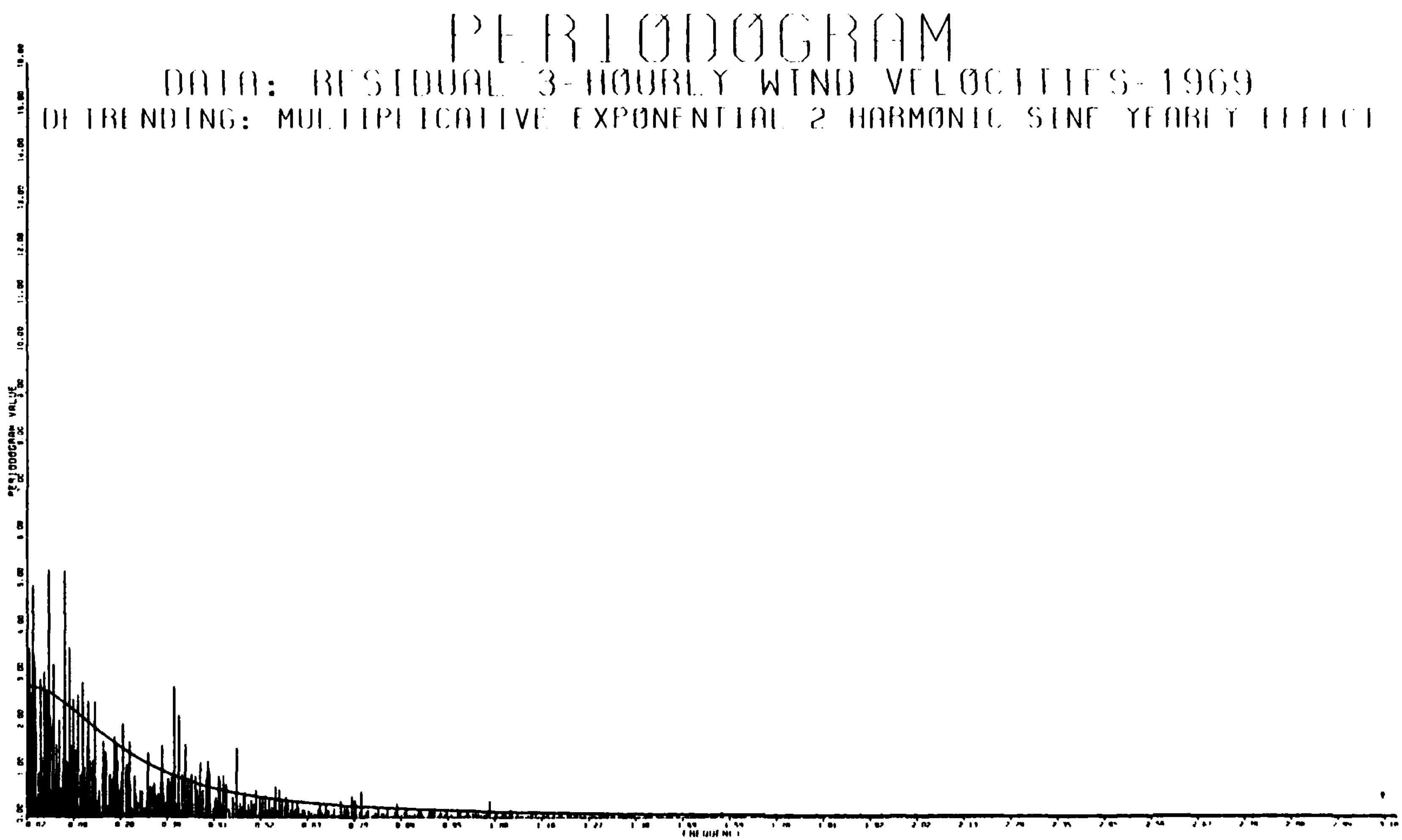

PERIODOGRAM
DATA: RESIDUAL 3-HOURLY WIND VELOCITIES-1969
DETRENDING: MULTIPLICATIVE EXPONENTIAL 2 HARMONIC SINE YEARLY EFFECT
PERIODOGRAM VALUE
FREQUENCY

Figure IV.E.4.

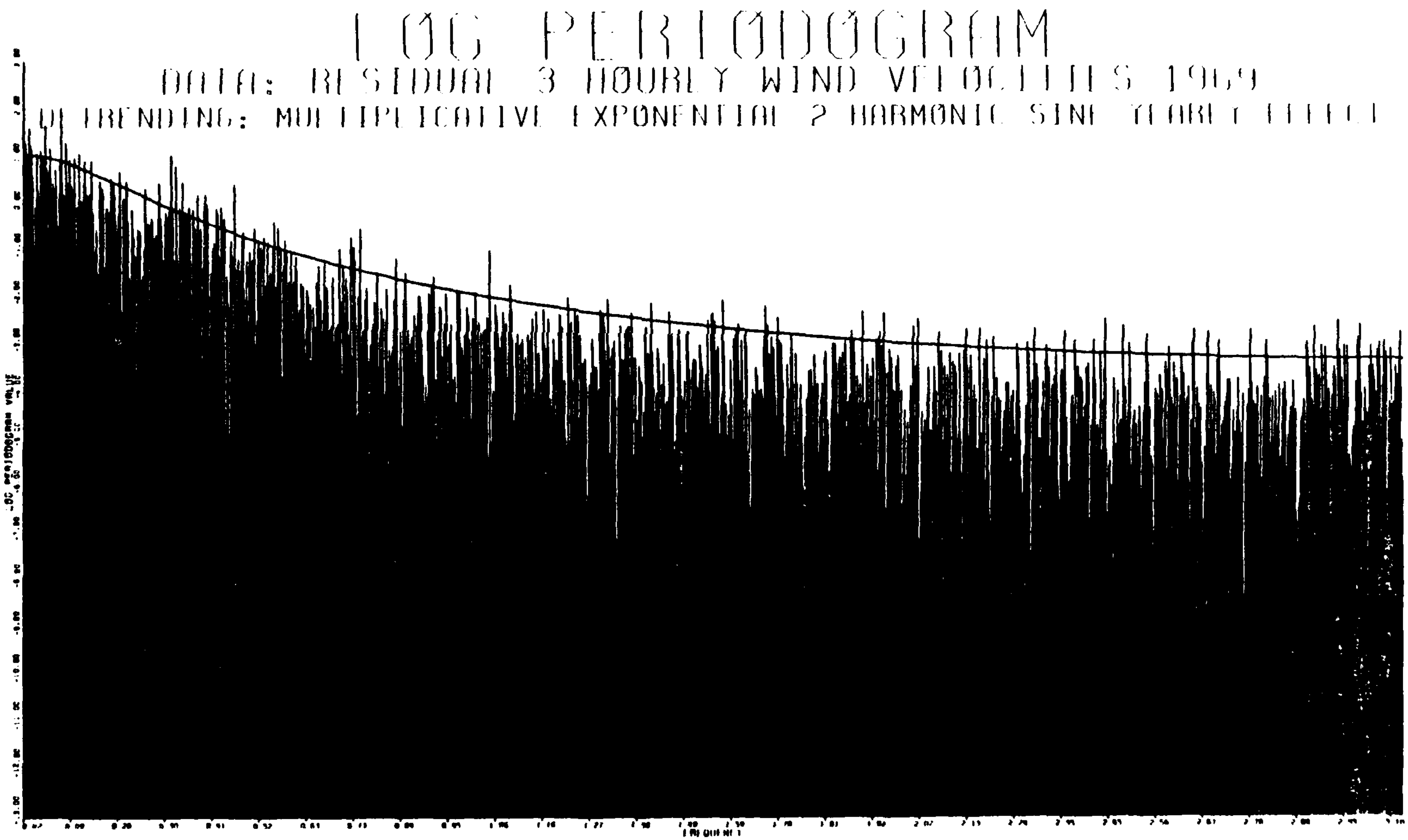

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 1969
DETRENDING: MULTIPLICATIVE EXPONENTIAL 2 HARMONIC SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.E.5.

(see equation IV.D.2). All of these plots show the yearly and six month cycles much reduced in importance. They also show some weak time-of-day effects, but these are more noticeable in the 1955 data.

The correlation structure of the data is shown in Table IV.E.1. The average data correlations are now lower than those of the average of the fifteen yearly correlations. They also drop more quickly than that of the average of the fifteen yearly correlations and eventually go below zero. This is another indication that the trend in the average data has been largely removed. Figures IV.E.6 and IV.E.7 show the periodogram and log periodogram for the averaged data. As has been noted previously, the time of day effects are more noticeable in the averaged data than they are in the data for a single year. However, the effects are prominent enough to warrent further consideration. This subject will be addressed in Section IV.G.

F. RESIDUAL ANALYSIS

Since a first-order autoregressive model appears to be a good fit to the innovation process $\{\varepsilon_n\}$, we need to examine this hypothesis more clearly. If we were dealing with a linear AR(1) model for the residual process

$$\varepsilon_n = \rho\varepsilon_{n-1} + Y_n \qquad \text{(IV.F.1)}$$

where Y_n is a sequence of iid random variables, then computing

Table IV.E.1.

DATA: 3-HOURLY WIND VELOCITIES; DETRENDING: 2 HARMCNIC EXPONENTIAL SINE

LAG	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	15 YR AVG	15 YR SD	AVG CATA
1	0.81	0.80	0.81	0.82	0.83	0.79	0.81	0.81	0.83	0.80	0.84	0.83	0.82	0.87	0.84	0.821	0.019	0.794
2	0.67	0.66	0.67	0.69	0.70	0.66	0.67	0.69	0.69	0.66	0.71	0.69	0.69	0.75	0.71	0.686	0.023	0.653
3	0.54	0.53	0.54	0.57	0.58	0.54	0.53	0.56	0.57	0.54	0.58	0.56	0.56	0.63	0.58	0.561	0.025	0.515
4	0.42	0.44	0.45	0.49	0.49	0.44	0.43	0.47	0.47	0.44	0.48	0.48	0.48	0.51	0.47	0.463	0.025	0.433
5	0.32	0.35	0.36	0.41	0.40	0.35	0.33	0.39	0.37	0.35	0.38	0.39	0.39	0.42	0.37	0.371	0.027	0.325
6	0.25	0.29	0.31	0.36	0.33	0.29	0.25	0.33	0.30	0.27	0.30	0.33	0.35	0.34	0.30	0.307	0.033	0.271
7	0.19	0.24	0.26	0.33	0.28	0.25	0.18	0.27	0.26	0.21	0.25	0.27	0.29	0.28	0.24	0.252	0.038	0.216
8	0.14	0.20	0.24	0.30	0.25	0.21	0.13	0.25	0.23	0.19	0.21	0.23	0.26	0.24	0.21	0.220	0.043	0.198
9	0.10	0.17	0.20	0.27	0.22	0.18	0.08	0.20	0.19	0.15	0.17	0.17	0.21	0.21	0.19	0.180	0.044	0.136
10	0.08	0.15	0.18	0.23	0.19	0.14	0.05	0.17	0.16	0.11	0.15	0.15	0.19	0.19	0.18	0.155	0.044	0.110
11	0.08	0.14	0.16	0.21	0.16	0.11	0.04	0.14	0.14	0.08	0.13	0.11	0.14	0.17	0.18	0.134	0.043	0.073
12	0.09	0.15	0.17	0.21	0.16	0.09	0.04	0.14	0.13	0.07	0.12	0.10	0.13	0.17	0.18	0.129	0.044	0.078
13	0.09	0.14	0.16	0.20	0.15	0.06	0.02	0.14	0.12	0.06	0.10	0.07	0.10	0.16	0.18	0.115	0.049	0.048
14	0.09	0.14	0.16	0.20	0.14	0.05	0.02	0.14	0.10	0.05	0.10	0.05	0.09	0.15	0.17	0.110	0.051	0.048
15	0.08	0.14	0.14	0.19	0.14	0.03	0.02	0.14	0.10	0.05	0.09	0.03	0.07	0.15	0.17	0.102	0.053	0.032
16	0.08	0.15	0.15	0.19	0.14	0.03	0.03	0.14	0.09	0.06	0.09	0.03	0.07	0.15	0.16	0.104	0.052	0.042
17	0.06	0.13	0.14	0.17	0.13	0.02	0.01	0.12	0.08	0.05	0.08	0.00	0.06	0.16	0.14	0.090	0.054	-.001
18	0.04	0.14	0.13	0.16	0.13	0.01	0.00	0.11	0.06	0.04	0.09	0.00	0.06	0.15	0.13	0.084	0.055	[illegible]
19	0.03	0.13	0.12	0.15	0.12	-.01	0.00	0.11	0.06	0.04	0.10	-.01	0.04	0.15	0.11	0.076	0.056	[illegible]
20	0.02	0.12	0.13	0.15	0.12	-.01	0.00	0.11	0.06	0.04	0.11	-.01	0.05	0.15	0.09	0.075	0.055	-.038
21	0.00	0.11	0.12	0.13	0.10	-.02	-.01	0.09	0.06	0.02	0.11	-.01	0.06	0.15	0.07	0.065	0.055	-.066
22	0.00	0.11	0.13	0.13	0.10	-.02	-.02	0.07	0.06	0.03	0.11	-.01	0.08	0.14	0.05	0.065	0.054	-.061
23	0.01	0.11	0.11	0.13	0.10	-.01	-.03	0.05	0.07	0.03	0.11	-.01	0.08	0.13	0.05	0.062	0.051	-.071
24	0.01	0.11	0.11	0.12	0.10	-.00	-.02	0.06	0.08	0.04	0.11	0.00	0.10	0.11	0.03	0.065	[illegible]	-.043
25	0.01	0.10	0.10	0.12	0.09	-.01	-.03	0.03	0.08	0.02	0.11	-.00	0.08	0.10	0.02	0.054	[illegible]	-.054
26	0.00	0.11	0.09	0.10	0.10	-.01	-.03	0.02	0.07	0.01	0.11	0.00	0.07	0.08	0.01	0.049	0.046	-.048
27	-.01	0.09	0.07	0.10	0.09	0.00	-.03	0.02	0.06	0.00	0.10	-.01	0.05	0.07	0.0	0.041	0.043	-.050
28	-.01	0.08	0.07	0.09	0.10	0.01	-.02	0.02	0.05	0.01	0.10	-.00	0.05	0.07	-.00	0.041	0.041	-.016
29	-.00	0.07	0.05	0.09	0.09	0.01	-.02	0.01	0.03	0.01	0.09	0.00	0.03	0.07	0.00	0.035	0.036	-.024
30	0.00	0.07	0.05	0.07	0.10	0.02	-.03	0.01	0.01	0.00	0.08	0.01	0.03	0.08	0.01	0.035	0.037	0.0
31	0.01	0.07	0.04	0.07	0.11	0.01	-.02	0.00	0.00	-.00	0.07	0.01	0.02	0.10	0.01	0.034	0.040	-.009
32	0.03	0.08	0.05	0.08	0.11	0.01	-.00	0.02	0.01	0.00	0.06	0.02	0.04	0.13	0.02	0.044	0.040	0.015

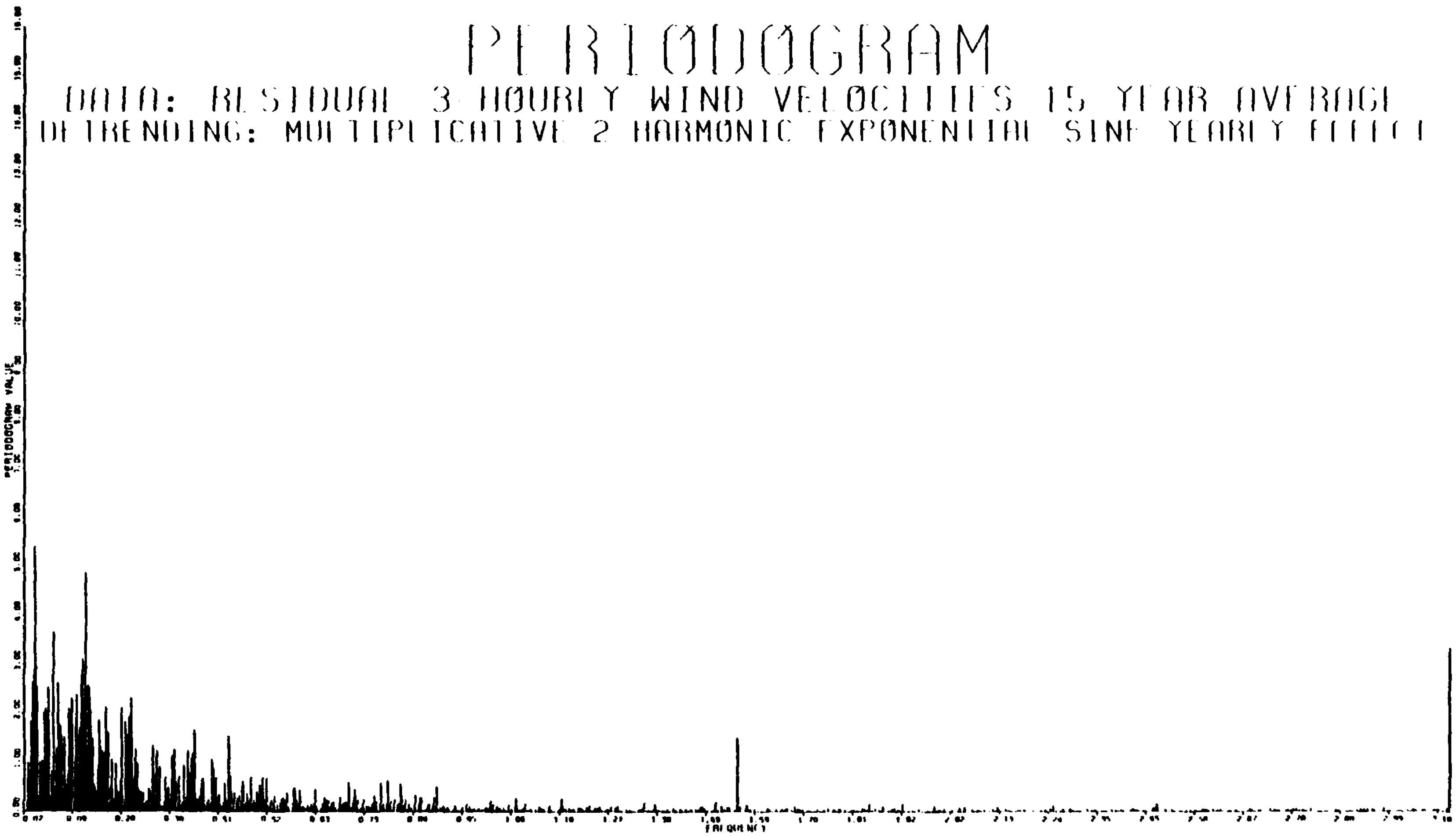


Figure IV.E.6. Prominent time of day cycles are evident in the average data after 2 harmonic detrending.

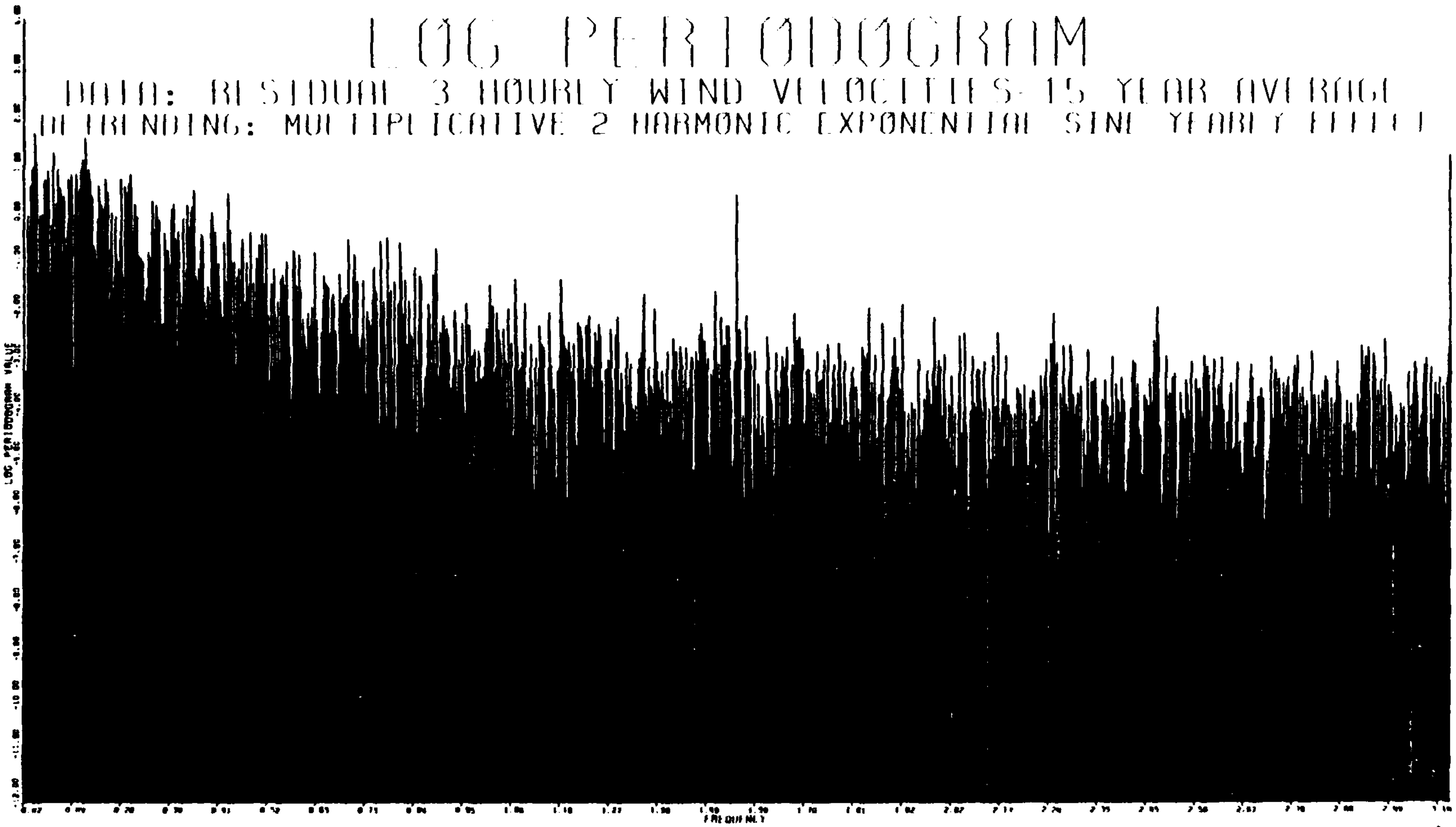

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES-15 YEAR AVERAGE
DETRENDING: MULTIPLICATIVE 2 HARMONIC EXPONENTIAL SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.E.7.

$$\varepsilon_n - \hat{\rho}\varepsilon_{n-1} = \hat{Y}_n \qquad \text{(IV.F.2)}$$

as an estimate of Y_n would be of interest. The estimated $\{Y_n\}$ should have a flat spectrum. Using the Gamma generation scheme of equation II.A.5 (i.e., $\varepsilon_n = B_n\varepsilon_{n-1} + C_nG_n$) reduces the value of differencing since the coefficients in the generation scheme, B_n and C_n, are continuous random variables and not constants. However, this differencing procedure may produce some insight to the data. Hence, the differences

$$\hat{Y}_n = \varepsilon_n - \hat{\rho}(1)\varepsilon_{n-1} \qquad \text{(IV.F.3)}$$

were produced, where ρ is a one-step (lag one) correlation for the two harmonic detrended data and ε_n and ε_{n-1} are two harmonic detrended data values.

Since the data has been detrended and the differencing serves to remove the dependence from the data, one would expect the periodogram of the detrended, difference data to be flat. The periodogram and log periodogram for the detrended, differenced data are presented in Figures IV.F.1 and IV.F.2. With the exception of a relatively strong indication of a six and twelve hour cycle, the periodogram is, in fact, reasonably flat. The log periodogram indicates the same characteristics.

The correlogram for the detrended, differenced data is Figure IV.F.3. There are two key points. First, all the correlations

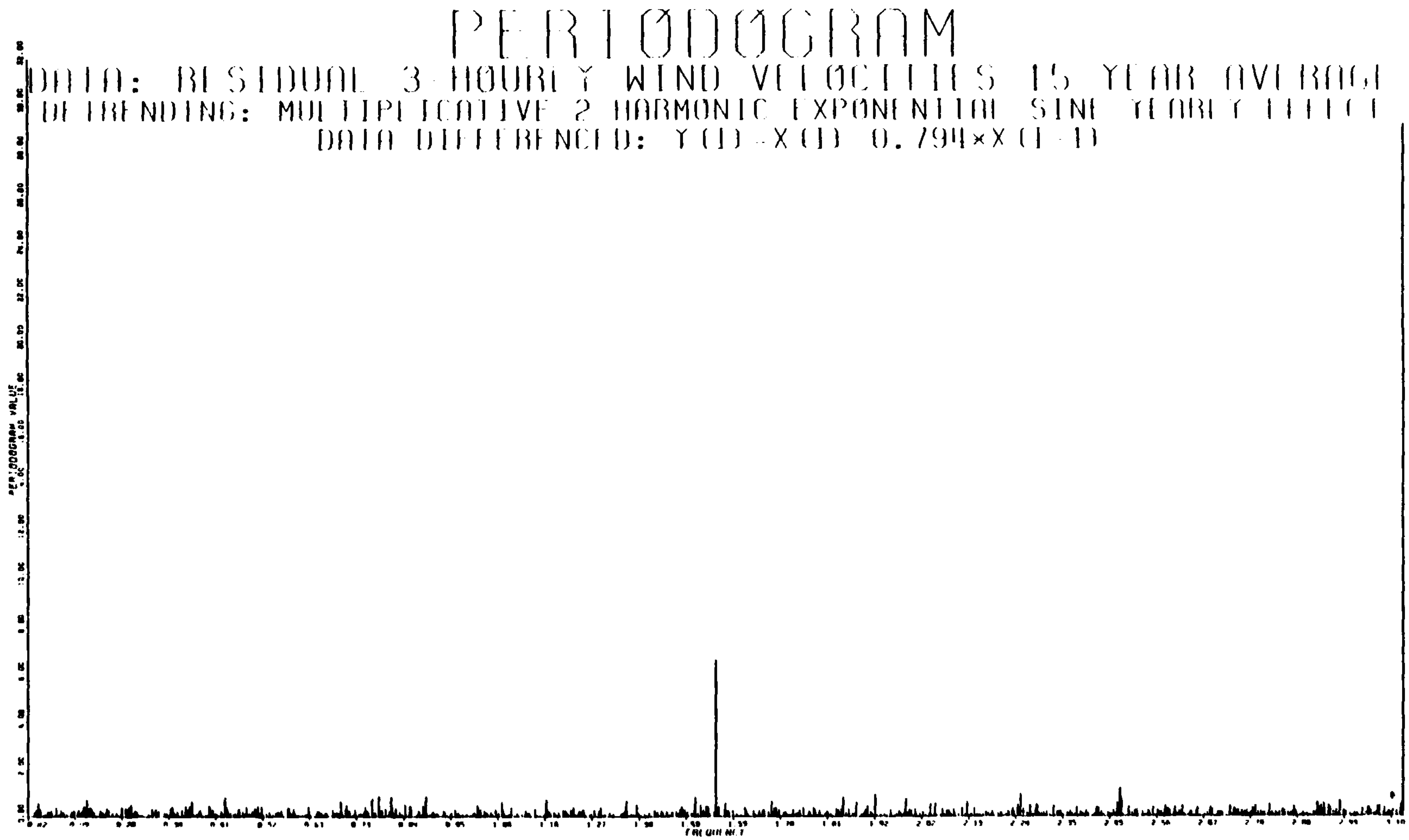


Figure IV.F.1. Data prewhitened to remove the autoregressive component shows presence of six and twelve hour cycles.

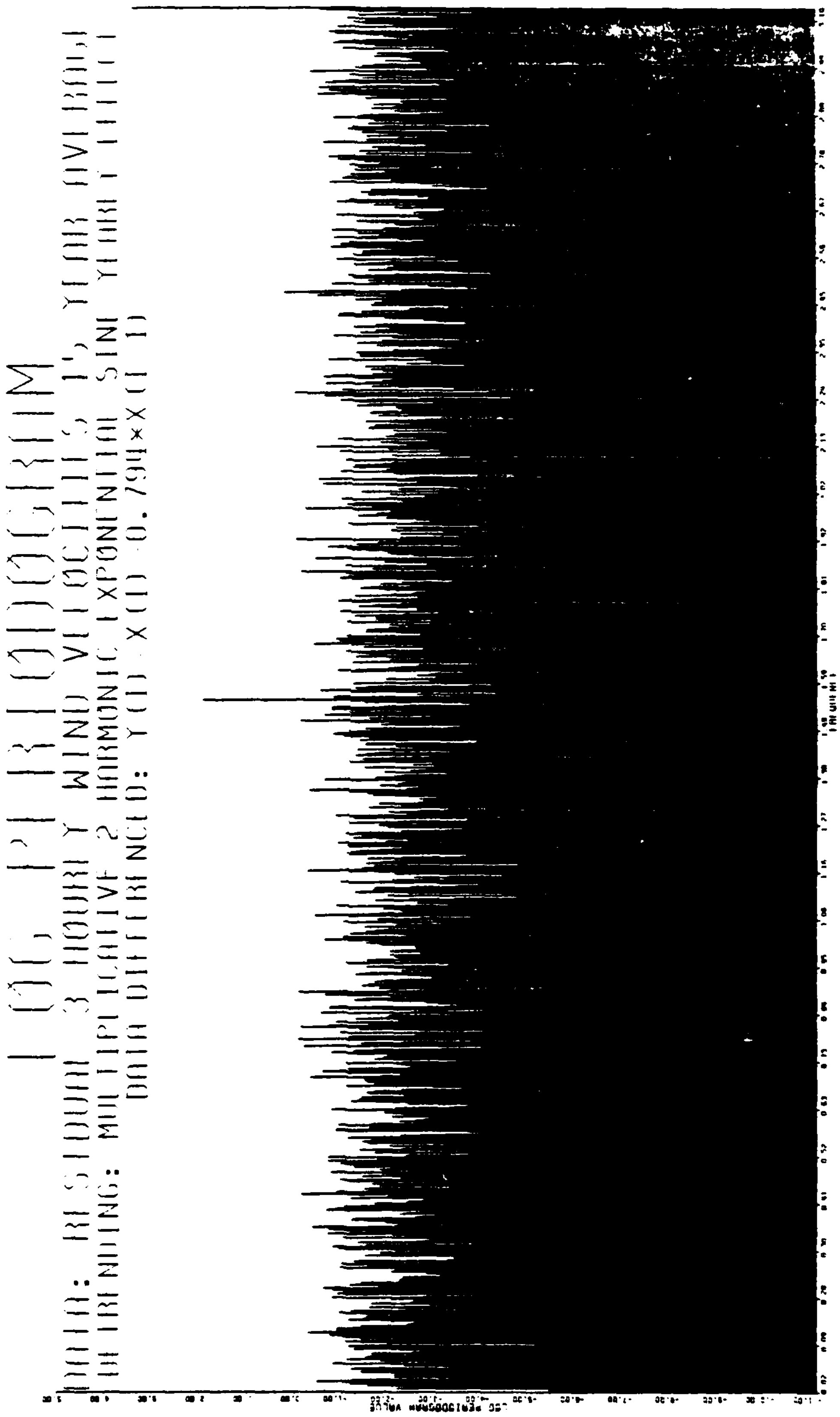
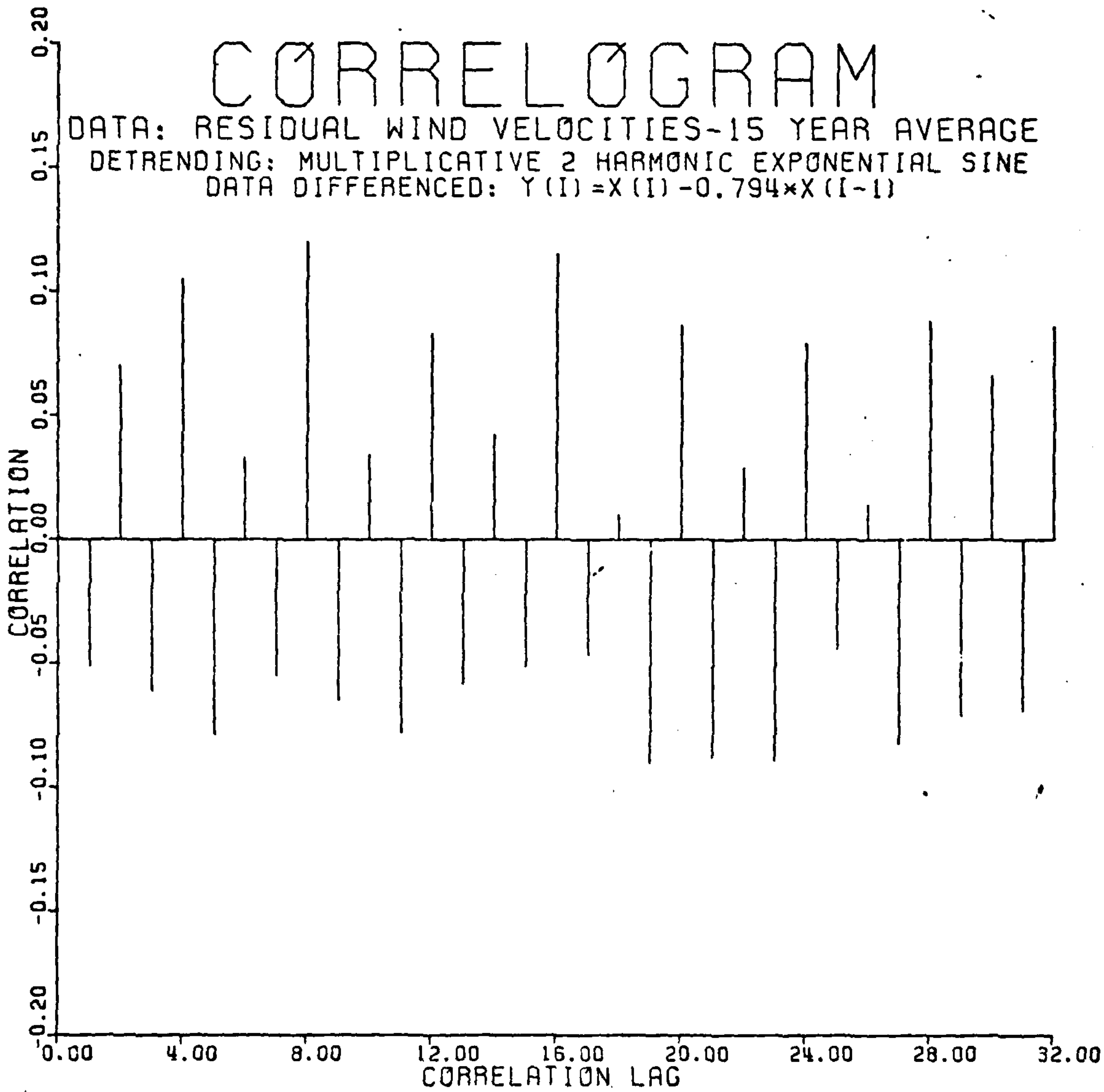

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 15 YEAR AVERAGE
DETRENDING: MULTIPLICATIVE 2 HARMONIC EXPONENTIAL SINE YEARLY EFFECT
DATA DIFFERENCED: Y(I) = X(I) - 0.794*X(I-1)
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.F.2.



CØRRELØGRAM
DATA: RESIDUAL WIND VELOCITIES-15 YEAR AVERAGE
DETRENDING: MULTIPLICATIVE 2 HARMONIC EXPONENTIAL SINE
DATA DIFFERENCED: Y(I)=X(I)-0.794*X(I-1)
CORRELATION
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
0.00
4.00
8.00
12.00
16.00
20.00
24.00
28.00
32.00
CORRELATION LAG

FIGURE IV. F.3

are relatively low, indicating that the dependence structure has been largely removed. Second, the alternation of the sign of the correlations is an indication that there still exists an important cyclic component in the data, in particular an alternation of twelve, or six hours. Differencing two sine functions (i.e., $\sin(\frac{2\pi n}{N}) - \sin(\frac{2\pi(n+1)}{N})$ will produce a cycle with period of two if they have non-zero amplitude. Therefore, the alternation of the correlations is evidence that an important cycle still remains in the data.

G. A FURTHER REFINEMENT OF THE MEAN; THE LAST DETRENDING

Since the evidence of the preceding two sections suggests that there is still one important cycle in the data, a further refinement of the model for the mean was undertaken. The evidence suggests that there may be six and twelve hour cycles in the data. These cycles may be the result of the passage of pressure fronts over the data collection location.

Only the exponential sine model for the mean was considered in the final detrending. The final model for the mean was

$$\mu_n = \mathrm{EXP}[a+b_1\sin(\frac{2\pi n}{2920}) + b_2\cos(\frac{2\pi n}{2920}) + b_2\sin(\frac{2\pi n}{1460}) + b_4\cos(\frac{2\pi n}{1460}) + b_4\sin(\frac{2\pi n}{4}) + b_6\cos(\frac{2\pi n}{4}) + b_7\cos(\frac{2\pi n}{2})] \qquad \text{(IV.G.1)}$$

One should note that the sine function with a period of two is omitted from the model. This is because the sin $(n\pi)$ is

identically zero if n is an integer. The implication of this is that we essentially lose the ability to determine the phase shift for the cycle with period two. This may mean that our attempt to remove the six hour cycle will not be completely successful. Parameters for the model in IV.G.1 were produced by the same techniques used previously (see Section IV.C and Table IV.C.1).

Figures IV.G.1 and IV.G.2 show the periodogram and log periodogram for 1955 data after detrending using the model of the mean in IV.G.1 (see also Table IV.C.1). With the exception of the six hour cycle, the periodogram compares favorably with the theoretical AR(1) periodogram superimposed over it. The log periodogram shows the same characteristics. Similar information is presented for 1969 in Figures IV.G.3 and IV.G.4. The strength of the six hour cycle is reduced for this year. Finally, the periodogram and log periodogram for the averaged data are presented in Figures IV.G.5 and IV.G.6. The comparison of the averaged data with a theoretical AR(1) process is considered acceptable.

Note too that in Table IV.G.1 the 15 year average correlation is commensurate with the correlation computed from the averaged data. Thus the discrepancy between these quantities noted in Table IV.B.2 has been explained.

It may be worth noting in passing that a surprising result of this analysis is the failure to detect any multiple-day cycles. Apparently some meteorologists believe that there

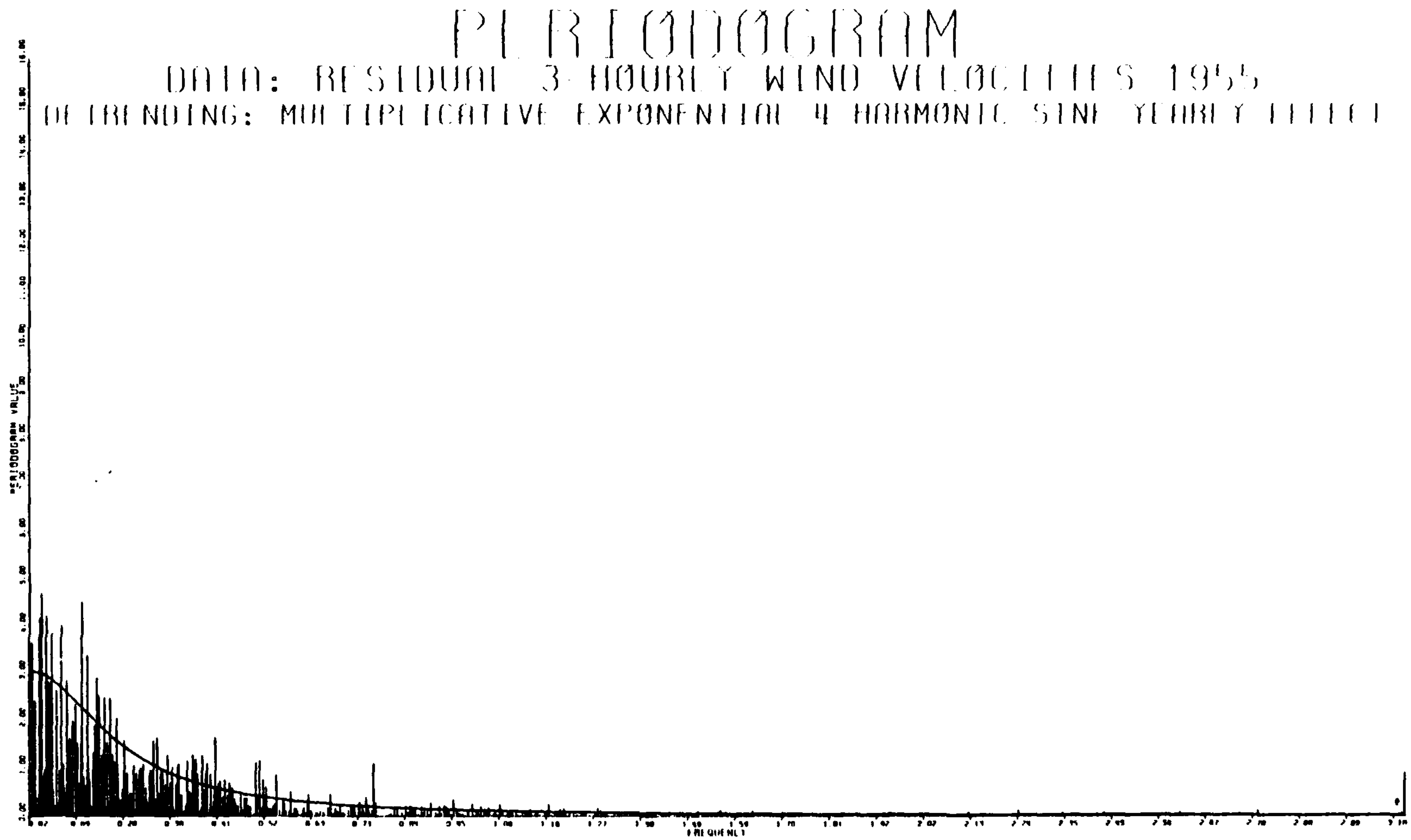
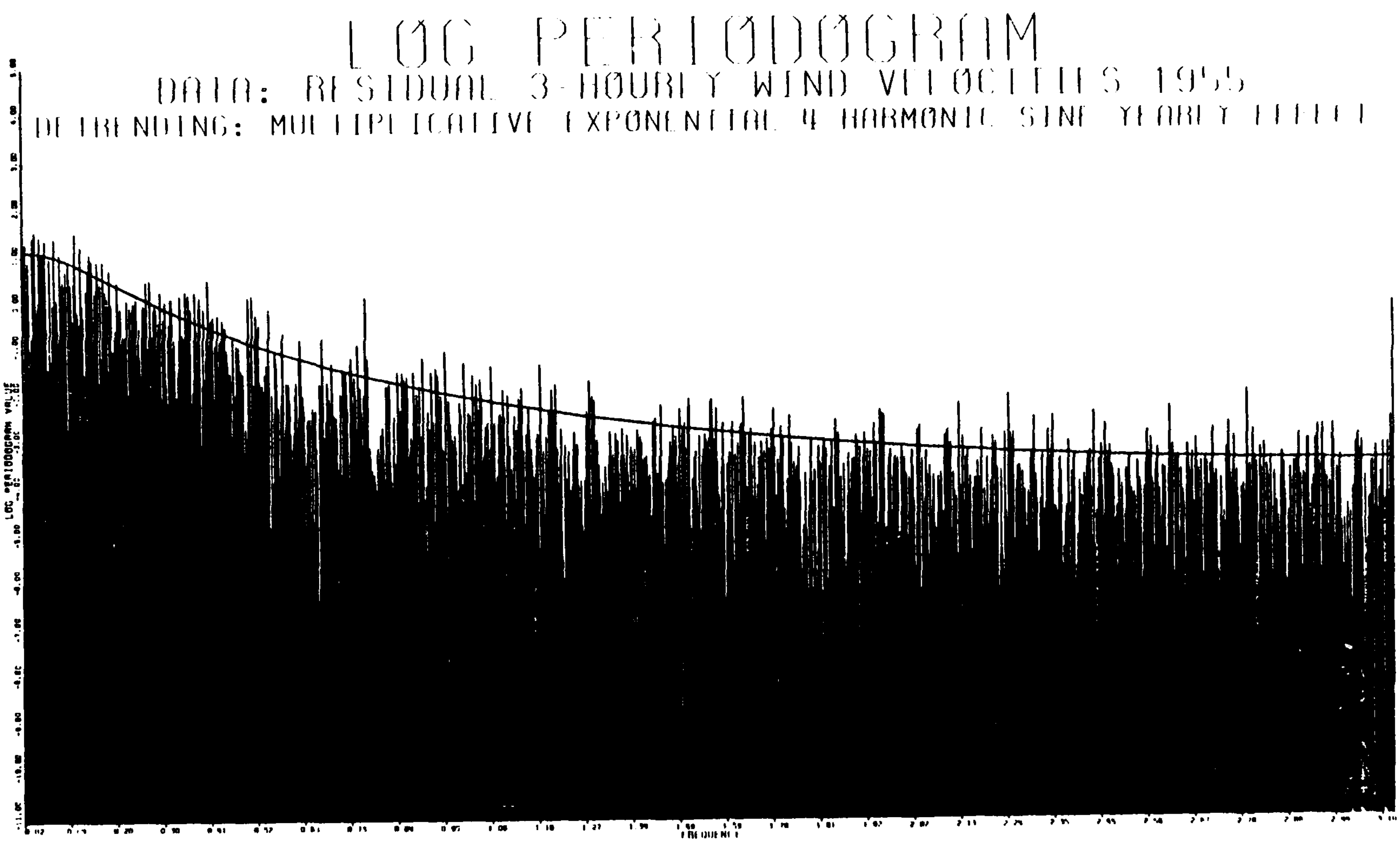


Figure IV.G.1. Presence of six hour cycle after 4 harmonic detrending indicates some heterogeneity among years.



LOG PERIODOGRAM
DATA: RESIDUAL 3-HOURLY WIND VELOCITIES 1955
DETRENDING: MULTIPLICATIVE EXPONENTIAL 4 HARMONIC SINE YEARLY EFFECT
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.G.2.

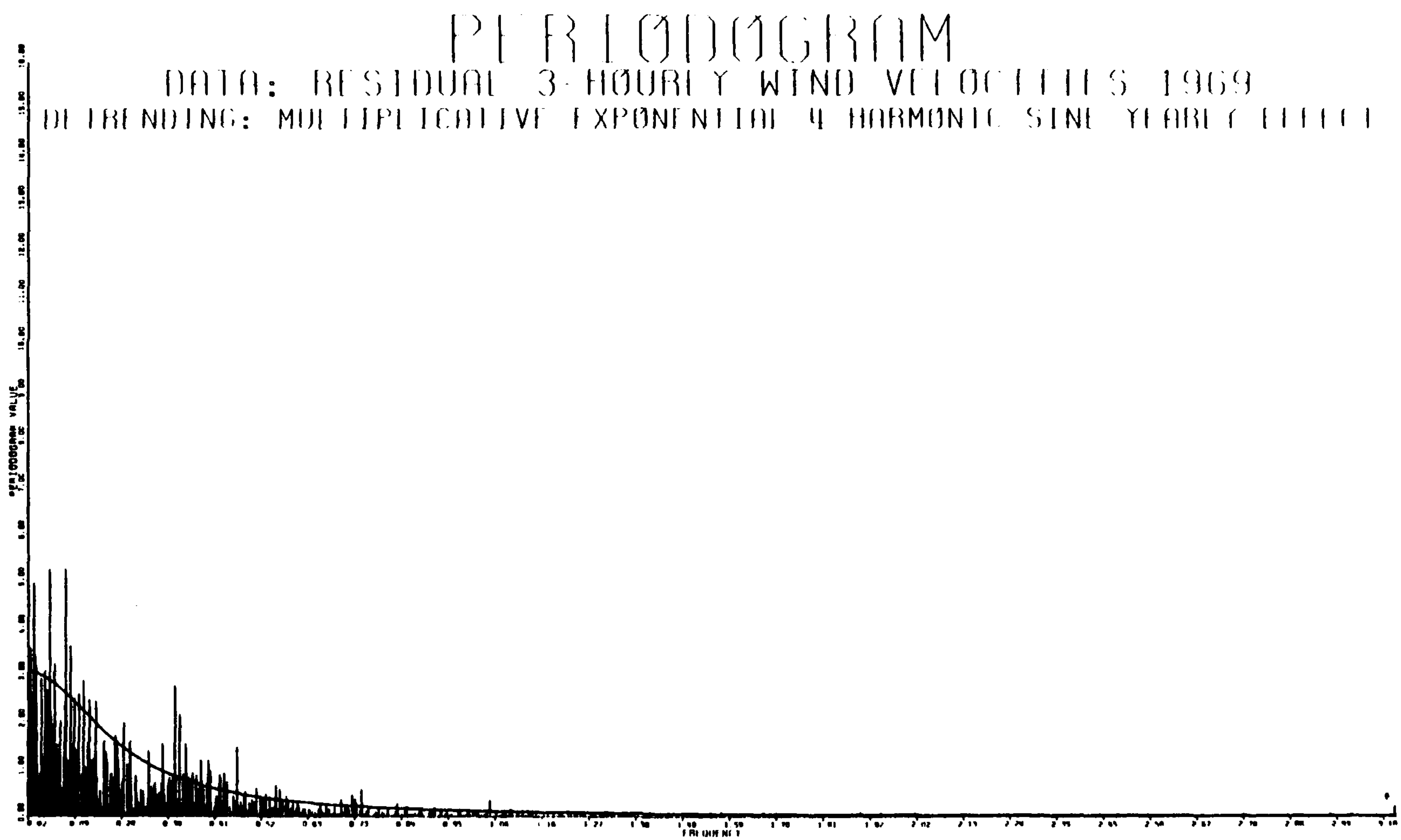

PERIODOGRAM
DATA: RESIDUAL 3-HOURLY WIND VELOCITIES 1969
DETRENDING: MULTIPLICATIVE EXPONENTIAL 4 HARMONIC SINE YEARLY EFFECT
PERIODOGRAM VALUE
FREQUENCY

Figure IV.G.3.

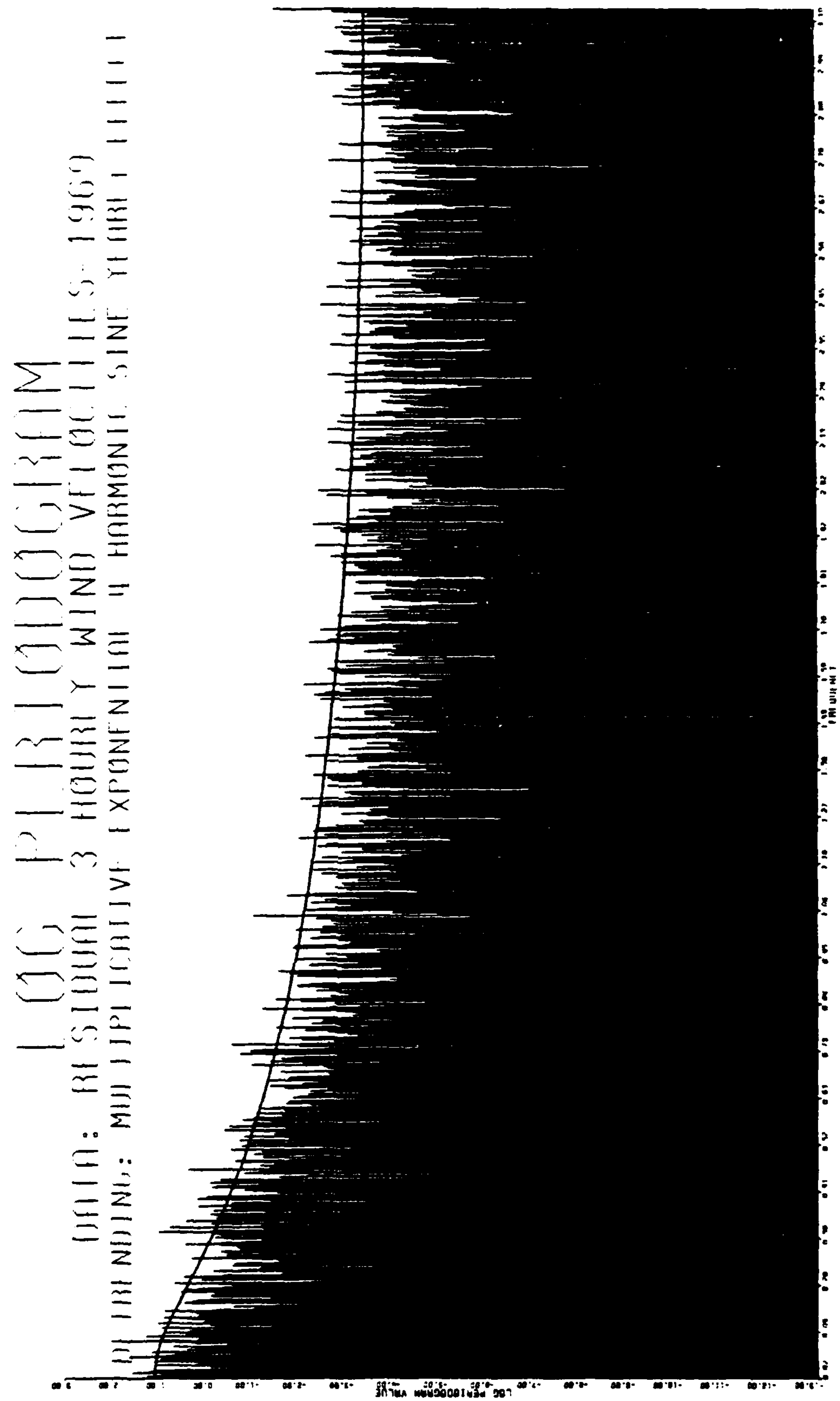

LOG PERIODOGRAM
DATA: RESIDUAL 3 HOURLY WIND VELOCITIES 1969
LOG PERIODOGRAM VALUE
FREQUENCY

Figure IV.G.4.

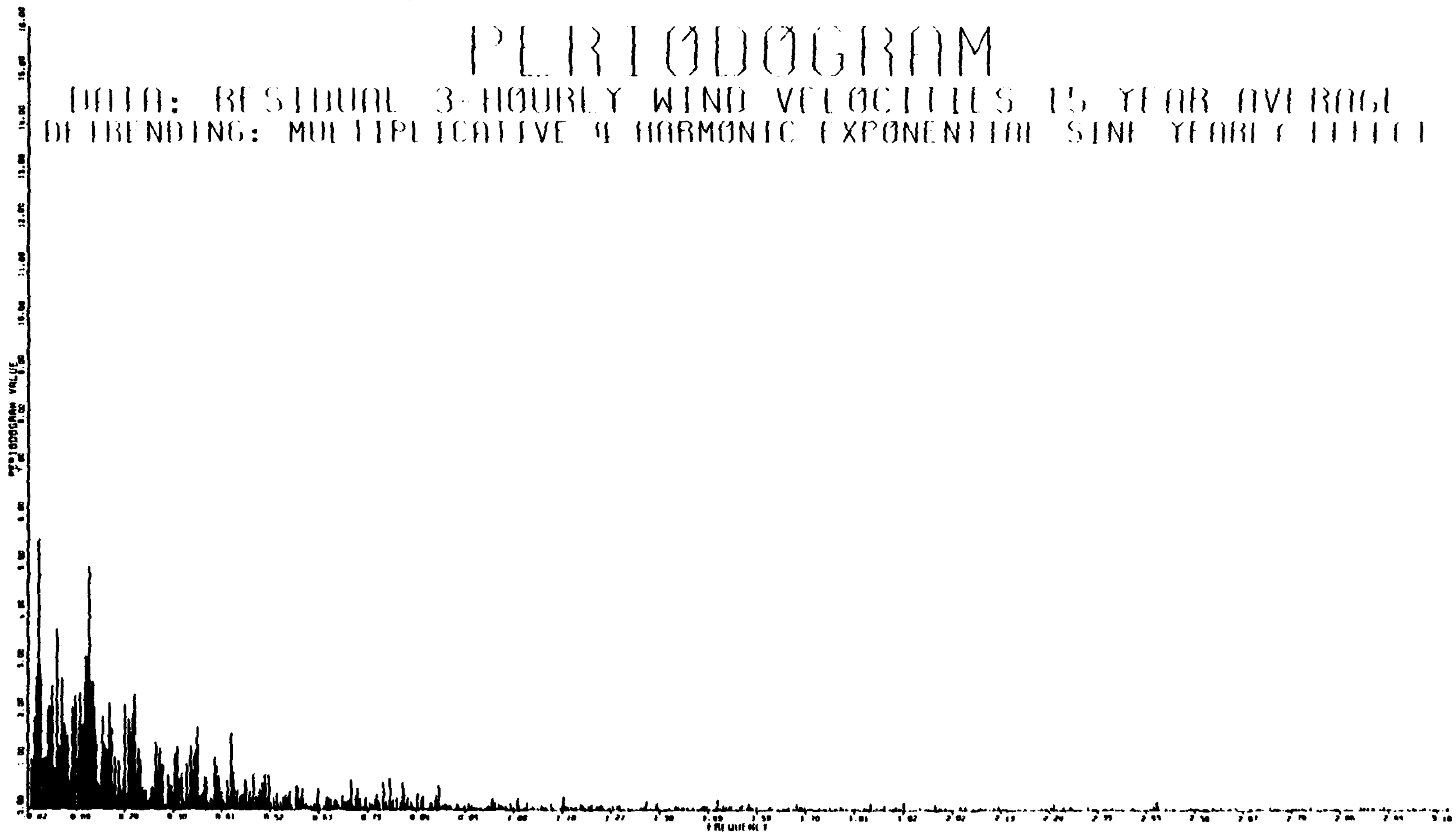


Figure IV.G.5. Average data shows autoregressive behavior with *no* cycles after 4 harmonic detrending.

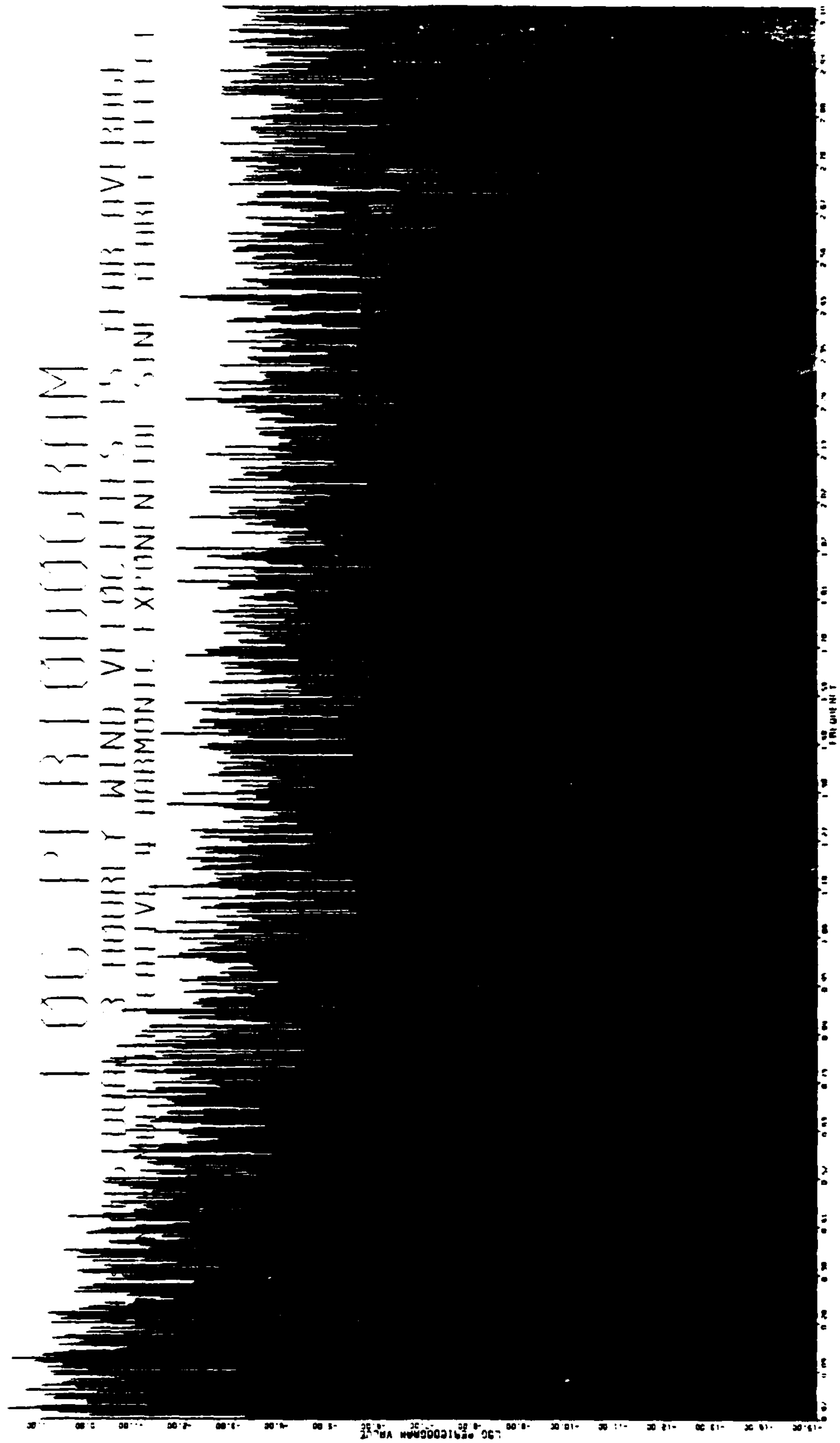


Figure IV.G.6.

is a six-day weather-cycle driven by the passage of storms. This analysis has failed to detect any such cycle. It may be that the high correlation among the data and the expectation that the actual storm cycle will be reflected in the data has created an impression that these cycles exist in the data when, in fact, they do not. This confusion of quasi-cycles produced by high positive correlation and completely deterministic cycles is common in applied science. Figure IV.G.7 shows a sample path for a GLAR(1) process with high correlation, $\rho(1) = 0.85$. Although it may be tempting to conclude that this process is showing evidence of a cyclic nature, there is no deterministic cycle in the data shown. The behavior displayed in this figure is typical of an autoregressive process with high correlation, and no cycle.

A table of correlations for the 4 harmonic detrended data is provided in Table IV.G.1. Its characteristics are much the same as those of the two harmonic detrended data.

H. SUMMARY

The model suggested for the representation of the wind speed data now has the following form. The basic structure is that of a multiplicative model, that is it has the form

$$X_n = \mu_n \varepsilon_n, \quad n = 1,2,\ldots \,. \tag{IV.H.1}$$

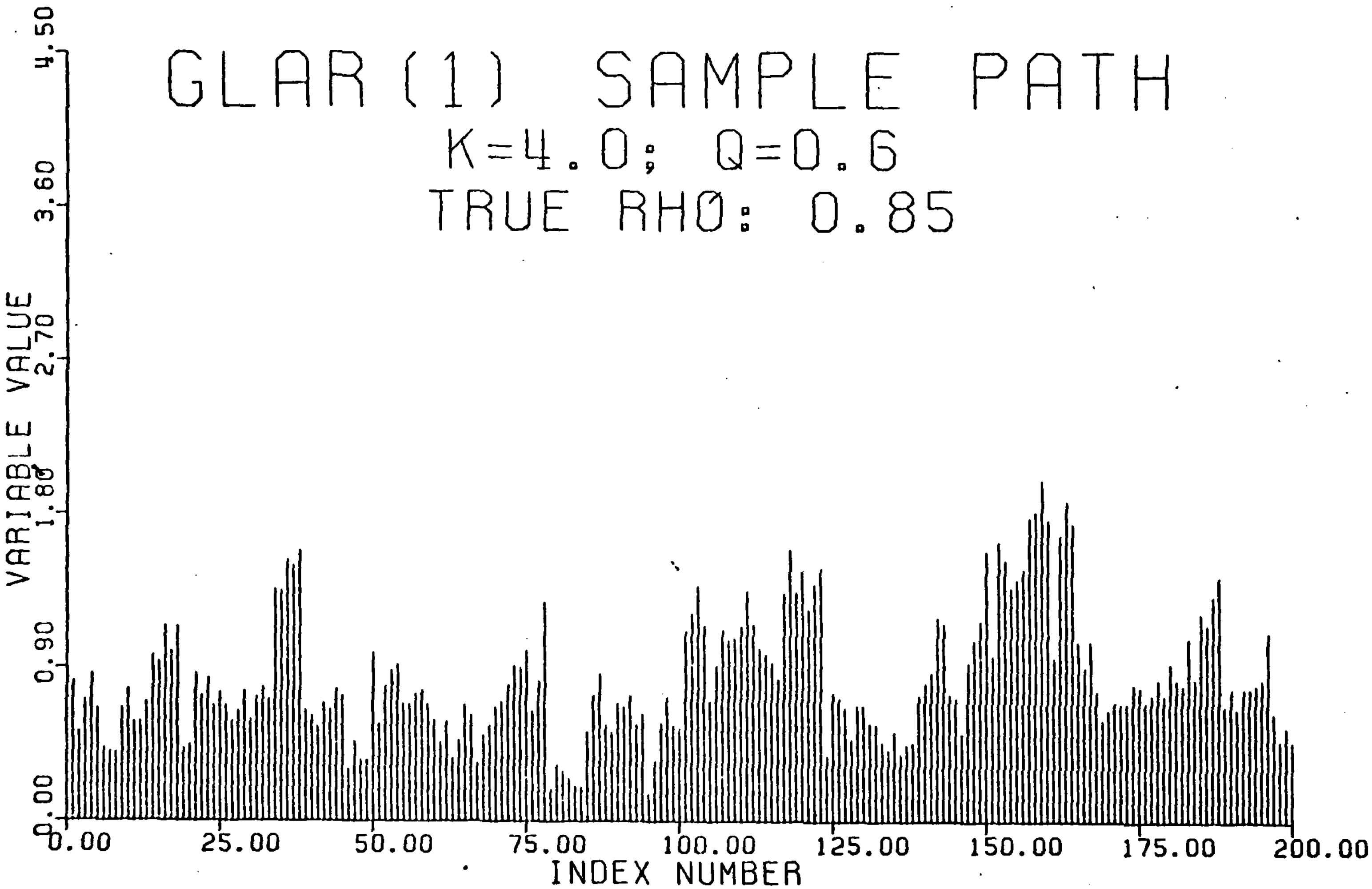


FIGURE IV. G.7 Sample Path Shows "Quasi Cyclic" Nature of an Autoregressive Process with High Correlation

Table IV.G.1.

DATA: 3-HOURLY WIND VELOCITIES; DETRENDING: 4 HARMONIC EXPONENTIAL SINE

LAG	1955	1956	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966	1967	1968	1969	15 YR AVG	15 YR SD	AVG CATA
1	0.81	0.79	0.81	0.82	0.83	0.80	0.81	0.82	0.83	0.81	0.84	0.83	0.82	0.87	0.84	0.822	0.019	0.812
2	0.67	0.66	0.67	0.68	0.70	0.66	0.67	0.69	0.69	0.66	0.71	0.69	0.69	0.75	0.71	0.686	0.023	0.661
3	0.53	0.53	0.54	0.57	0.59	0.54	0.53	0.56	0.57	0.54	0.58	0.56	0.57	0.63	0.57	0.562	0.025	0.530
4	0.42	0.44	0.45	0.49	0.49	0.44	0.43	0.47	0.47	0.44	0.47	0.48	0.48	0.51	0.47	0.463	0.025	0.426
5	0.32	0.35	0.36	0.41	0.40	0.35	0.33	0.39	0.38	0.35	0.38	0.39	0.40	0.42	0.37	0.372	0.028	0.337
6	0.25	0.29	0.31	0.36	0.33	0.29	0.25	0.33	0.30	0.27	0.30	0.33	0.35	0.34	0.30	0.307	0.032	0.273
7	0.19	0.23	0.26	0.33	0.28	0.25	0.18	0.28	0.26	0.21	0.25	0.27	0.30	0.28	0.24	0.253	0.038	0.226
8	0.14	0.20	0.24	0.31	0.25	0.21	0.13	0.25	0.23	0.18	0.21	0.23	0.26	0.24	0.21	0.219	0.044	0.187
9	0.10	0.16	0.20	0.27	0.22	0.18	0.09	0.20	0.20	0.15	0.17	0.17	0.21	0.21	0.19	0.181	0.044	0.146
10	0.08	0.15	0.18	0.23	0.18	0.14	0.05	0.17	0.16	0.11	0.15	0.15	0.19	0.19	0.18	0.155	0.043	0.112
11	0.08	0.14	0.16	0.22	0.16	0.11	0.04	0.14	0.14	0.08	0.13	0.12	0.15	0.17	0.18	0.135	0.043	0.084
12	0.09	0.15	0.17	0.21	0.16	0.08	0.03	0.14	0.13	0.07	0.12	0.10	0.13	0.17	0.18	0.128	0.045	0.065
13	0.09	0.14	0.16	0.20	0.15	0.06	0.02	0.14	0.12	0.06	0.10	0.07	0.10	0.16	0.18	0.116	0.049	0.056
14	0.09	0.14	0.16	0.20	0.14	0.05	0.02	0.14	0.10	0.05	0.10	0.05	0.09	0.15	0.17	0.110	0.050	0.048
15	0.08	0.13	0.14	0.19	0.14	0.03	0.02	0.14	0.10	0.05	0.09	0.03	0.07	0.15	0.16	0.103	0.052	0.041
16	0.08	0.15	0.15	0.19	0.14	0.03	0.02	0.14	0.09	0.06	0.09	0.03	0.07	0.15	0.16	0.103	0.053	0.030
17	0.05	0.13	0.14	0.17	0.13	0.02	0.01	0.13	0.08	0.05	0.08	0.01	0.06	0.16	0.14	0.090	0.054	0.008
18	0.04	0.15	0.13	0.16	0.13	0.01	0.00	0.12	0.06	0.04	0.09	0.00	0.05	0.15	0.13	0.084	0.055	-.022
19	0.02	0.13	0.12	0.15	0.12	-.00	0.00	0.12	0.06	0.04	0.10	-.01	0.05	0.15	0.11	0.077	0.056	-.042
20	0.02	0.13	0.13	0.15	0.11	-.01	0.0	0.11	0.06	0.03	0.11	-.01	0.05	0.15	0.09	0.074	0.055	-.051
21	0.00	0.11	0.12	0.13	0.10	-.02	-.01	0.09	0.06	0.02	0.11	-.01	0.06	0.15	0.07	0.066	0.055	-.060
22	0.00	0.12	0.13	0.13	0.10	-.02	-.02	0.07	0.06	0.03	0.11	-.01	0.08	0.14	0.05	0.065	0.054	-.061
23	0.00	0.11	0.11	0.13	0.10	-.00	-.03	0.05	0.07	0.03	0.11	-.01	0.09	0.13	0.05	0.062	0.051	-.064
24	0.01	0.11	0.11	0.12	0.10	-.01	-.02	0.05	0.08	0.03	0.11	0.00	0.10	0.11	0.03	0.064	0.049	-.057
25	0.01	0.10	0.10	0.12	0.09	-.01	-.02	0.03	0.08	0.02	0.11	0.00	0.08	0.10	0.02	0.054	0.048	-.047
26	0.00	0.11	0.09	0.10	0.10	-.01	-.03	0.02	0.07	0.01	0.11	0.00	0.07	0.08	0.01	0.049	0.046	-.049
27	-.01	0.09	0.07	0.10	0.09	0.00	-.03	0.02	0.06	0.01	0.10	-.01	0.05	0.07	-.00	0.041	0.043	-.042
28	-.01	0.09	0.07	0.09	0.10	0.01	-.03	0.02	0.05	0.01	0.10	-.00	0.05	0.07	-.00	0.040	0.042	-.030
29	-.01	0.07	0.05	0.08	0.09	0.01	-.02	0.01	0.03	0.01	0.09	0.00	0.04	0.07	0.00	0.035	0.036	-.016
30	0.00	0.07	0.05	0.07	0.10	0.02	-.02	0.01	0.01	0.00	0.08	0.01	0.03	0.09	0.01	0.035	0.037	-.002
31	0.01	0.07	0.04	0.08	0.11	0.01	-.02	0.00	0.01	-.00	0.07	0.01	0.03	0.10	0.01	0.034	0.040	-.001
32	0.03	0.09	0.05	0.08	0.11	0.01	-.01	0.02	0.00	-.00	0.06	0.02	0.03	0.13	0.02	0.043	0.040	0.003

The $\{X_n\}$ sequence represents the raw wind speed data. The μ_n is a deterministic function of n. The innovative terms $\{\varepsilon_n\}$ are modeled by a GLAR(1) process.

The GLAR(1) process was discussed and analyzed in Section II.B. The generation scheme presented in equation II.B.1.1 is repeated here (with ε_i replacing X_i).

$$\varepsilon_n = A_n \varepsilon_{n-1} + B_n G_n. \qquad \text{(IV.H.2)}$$

The innovative sequence $\{\varepsilon_n\}$ is itself correlated. The parameters of the GLAR(1) process control the correlation structure of the model. (See Section II.B.2, in particular equation II.B.2.1.)

The mean μ_n has been modeled as a four harmonic exponential sine function,

$$\mu_n = \text{EXP}[a+b_1\sin(\tfrac{2\pi n}{2920})+b_2\cos(\tfrac{2\pi n}{2920})+b_3\sin(\tfrac{2\pi n}{1460})+b_4\cos(\tfrac{2\pi n}{1460}) +b_5\sin(\tfrac{2\pi n}{4})+b_6\cos(\tfrac{2\pi n}{4})+b_7\cos(\tfrac{2\pi n}{2})] \qquad \text{(IV.H.3)}$$

The four harmonics represented are a yearly cycle (coefficients b_1 and b_2), a six month cycle (coefficients b_3 and b_4), a twelve hour cycle (coefficients b_5 and b_6) and a six hour cycle (coefficient b_7). The values for these parameters and the "a" parameter are found in Table IV.C.1.

The innovative terms are modeled by the GLAR(1) process. The parameter values for k and q were determined to be 2.843

and 0.727, respectively, by using the numerical approximation to the maximum likelihood method described in Section II.B.4. The data used for this evaluation were the residuals produced by the single harmonic exponential sine model of the mean. The parameters were not recomputed for the two or four harmonic exponential sine model because of time limitations.

These parameter values give a correlation of 0.744. This is somewhat less than the average correlation of 0.826 for the single harmonic residuals (see Table IV.E.1). However, this deviation is not considered serious. This is because the estimates produced by the four harmonic detrended data may differ from those produced by the two harmonic data and the correlations for the one harmonic data are modified by the presence of the six month, twelve hour and six hour cycles.

The simulation study of Section II.E indicated that for large k and high correlation the standard deviation of the maximum likelihood estimates was about half that for the moment estimates (see Figure II.E.1 and Table II.E.2). In addition, neither estimation procedure had any apparent bias. For these reasons the maximum likelihood estimates are preferred over the moment estimates in this case unless computer time is limited.

LIST OF REFERENCES

1. Box, G.E.P. and Jenkins, G.M., Time Series Analysis: Forecasting and Control, revised edition, Holden Day, 1976.

2. Gaver, D.P. and Lewis, P.A.W., "First-order autoregressive gamma sequences and point processes," Adv. Appl. Prob., 12, 727-745, 1980.

3. Feller, W., An Introduction to Probability Theory and Its Applications, Wiley, 1971.

4. Loeve, M., Probability Theory, 3rd ed., Van Nostrand, 1963.

5. Lawrance, A.J. and Lewis, P.A.W., "An exponential moving average sequence and point process (EMA1)," J. Appl. Prob., 14, 98-113, 1977.

6. Jacobs, P.A. and Lewis, P.A.W., "A mixed autoregressive-moving average exponential sequence and point process (EARMA(1,1)," Adv. Appl. Prob., 9, 87-104, 1977.

7. Lawrance, A.J. and Lewis, P.A.W., "The exponential auto-regressive-moving average process EARMA(p,q)," J. R. Statist. Soc. B, 1980.

8. Lawrance, A.J. and Lewis, P.A.W., "A new autoregressive time series model in exponential variables (NEAR(1)), Adv. Appl. Prob., 13, 824-845, 1981.

9. Jacobs, P.A., "A closed cyclic queuing network with dependent exponential service times," J. Appl. Prob., 15, 573-589, 1978.

10. Lewis, P.A.W. and Shedler, G.S., "Analysis and modeling of point processes in computer systems," Bull. ISI, XLVII(2), 193-210, 1978.

11. Lewis, P.A.W., "Simple multivariate time series for simulations of complex systems," Proceedings of 1981 Winter Simulation Conference, 389-390, 1981.

12. Naval Postgraduate School Technical Report NPS 55-78-022, Discrete time series generated by mixtures III: autoregressive processes (DAR(p)), by P.A. Jacobs and P.A.W. Lewis, September 1978.

13. Fishman, G.W., personal communication to Professor P.A.W. Lewis, 1982.

14. Lawrance, A.J. and Lewis, P.A.W., "Generation of some first-order autoregressive Markovian sequences of positive random variables with given marginal distributions," Proc. Appl. Prob./Computer Science Conference, R. Disney (ed.), 1982.

15. Schmeiser, to appear.

16. Naval Postgraduate School Technical Report NPS 55-81-005, The new Naval Postgraduate School random number package LLRANDOMII, by P.A.W. Lewis and L. Uribe, February 1981.

17. Quenouille, M.H., "Notes on bias in estimation," Biometrika, 43, 353-360, 1956.

18. Wold, H., "Sur les processes staionaires ponctuals," Colloques Internat, CNRSL 3, 75-86, 1948.

19. Cos, D.R., "Some statistical methods connected with series of events," J. R. Statist. Soc., B17, 129-164, 1955.

20. Marshall, A.W. and Olkin, I., "A multivariate exponential distribution," J. American Statist. Assoc., 62, 30-44, 1967.

21. McKenzie, E., "Extending the correlation structure of exponential autoregressive-moving average processes," J. Appl. Prob., 18, 181-189, 1981.

22. Tavares, L.V., "A non-Gaussian Markovian model to simulate hydrologic processes," Journal of Hydrology, 46, 281-287, 1980.

23. Chernick, M.R., Daly, D.J., and Littlejohn, R.P., "Concerning two Markov chains with exponential stationary distributions and charaterizations, to appear.

24. Gaver, D.P. and Lewis, P.A.W., "First-order autoregressive gamma sequences and point processes," Adv. Appl. Prob., 12, 727-745, 1980.

25. Moran, P.A.P., "Testing for correlation between non-negative variates," Biometrika, 54, 375-394, 1967.

26. Gaver, D.P., "Point process problems in reliability," Stochastic Point Processes: Statistical Analysis, Theory and Applications, P.A.W. Lewis (ed.), pp. 775-800, New York: Wiley, 1972.

27. Lawrance, A.J. and Lewis, P.A.W., "Dependent pairs of exponential and uniform random variables useful in simulation," to appear.

28. Chatfield, C., The Analysis of Time Series: An Introduction, 2d ed., London: Chapman and Hall.

29. Cox, D.R. and Lewis, P.A.W., The Statistical Analysis of Series of Events, Chapman and Hall, 1978.

30. Tavares, L.V., "An exponential Markovian stationary process," J. Appl. Prob., 17, 1117-1120, 1980.

31. Marshall, A.W. and Olkin, I., "A generalized bivariate exponential distribution," J. Appl. Prob., 4, 291-302, 1967.

32. Kotz, S. and Johnson, N.K., Continuous Univariate Distributions, Vol. 2, Houghton Mifflin, 1970.

33. Oceanographic Observation at Ocean Station P (50°N,145°W), 29 July-14 September, Pacific Marine Science Report 78-1, International Ocean Sciences, Offshore Oceanography Group, Sidney, B.C., Canada.

INITIAL DISTRIBUTION LIST

		No. Copies
1.	Defense Technical Information Center Cameron Station Alexandria, VA 22314	2
2.	Library, Code 0142 Naval Postgraduate School Monterey, CA 93940	2
3.	Department Chairman, Code 55 Department of Operations Research Naval Postgraduate School Monterey, CA 93940	1
4.	Professor P.A.W. Lewis, Code 55Lw Department of Operations Research Naval Postgraduate School Monterey, CA 93940	2
5.	Professor D.P. Gave, Code 55Gv Department of Operations Research Naval Postgraduate School Monterey, CA 93940	1
6.	Associate Professor P.A. Jacobs, Code 55Jc Department of Operations Research Naval Postgraduate School Monterey, CA 93940	1
7.	Assistant Professor R.W. Garwood, Code 68Gd Department of Oceanography Naval Postgraduate School Monterey, CA 93940	1
8.	Dr. A.J. Lawrance Department of Mathematical Statistics University of Birmingham Edgbaston, P.O. Box 363 Birmingham, England	1
9.	Professor R.R. Read, Code 55Re Department of Operations Research Naval Postgraduate School Monterey, CA 93940	1

10.	Associate Professor J.K. Hartman, Code 55Hh Department of Operations Research Naval Postgraduate School Monterey, CA 93940	1
11.	Professor R.J. Renard, Code 63Rd Department of Meteorology Naval Postgraduate School Monterey, CA 93940	1
12.	Operations Analysis Programs, Code 30 Naval Postgraduate School Monterey, CA 93940	2
13.	LTC David K. Hugus 7522 Monroe Ave. Hammond, IN 46324	1

DATE

FILMED

8